Real Options Theory for Real Asset Portfolios: the Oil Exploration Case

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Abstract

This paper discusses a *portfolio theory for real assets* with main focus on petroleum exploration and development assets. Exploratory assets are prospects with chances to find out development assets (oilfields in this case). By the real options point of view, exploratory assets are *compound options*. In opposition to financial assets portfolio theory, the paper shows that *positive correlation between exploratory assets is a desirable feature* because it increases both the (endogenous) *learning option* value and the *synergy* gain with development assets. In the first case due to the learning sequential nature, with the option to limit losses if occur bad news and creating the option to develop if occur good news. In the second case because a higher (positive) correlation increases the probability of multiple success and so the synergy gain by sharing the development infrastructure. The analysis of the simplest portfolio, i.e., with only two exploratory assets, provides important insights about learning, synergy and option to defer exploration. The optimal intertemporal distribution of projects shall use the concept of *option to defer*. A *necessary condition* for the immediate exercise of an exploratory option (wildcat drilling investment) is the existence of at least one scenario where the development option is deep-in-the-money. For all projects in which deferring is optimal, we need an idea of both the *probability of later exercise* and the *expected time of exercise*, conditional to option exercise occurrence. This portfolio planning is necessary for resource management purposes and is performed by *real-world* (and not risk-neutral) stochastic processes simulation. A multiple asset portfolio of exploratory prospects example is analyzed, highlighting the learning processes modeled as *information revelation processes*, with discussion of their properties.

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Keywords: real options, portfolio theory, petroleum exploration & production, real assets correlation, learning option, synergy, defer option, information revelation process, investment under uncertainty.

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1 – Introduction

The theory of financial portfolio is well developed and popular with the Nobel Laureate Markowitz' portfolio theory, which is based in the mean-variance optimization approach (see Markowitz, 1959). This theory highlights the *diversification* effect with the proposed optimization approach, so that we can reduce the risk (portfolio variance) without reducing the expected return, by choosing a suitable set of assets with low correlations between them. This theory has tentatively been extended to real assets portfolio case, mainly in professional literature. However, although there are good papers showing that diversification principles remain valid (e.g., see Ball Jr. & Savage, 1999), the real assets case demands a richer portfolio theory in order to capture issues like *synergy* between real assets and the real options embedded into real assets such as the *option to defer* and *learning* options.

A decade after the publication of first textbooks of Dixit & Pindyck (1994) and Trigeorgis (1996), the real options theory nowadays is well developed and widely accepted. However, the portfolio theory for real assets under uncertainty remains in its infancy even considering the large literature on real options, which has focused on single asset valuation (in some cases with multiple interacting options in the *same* asset). There are some exceptions, some of which are briefly discussed here (section 2).

This paper analyzes the portfolio theory for real assets with emphasis on the *role of correlation* on synergy and learning, with focus on the petroleum exploration portfolio case. This application allows simple examples to understand the role of *learning options* and *synergy* between two or more exploratory assets, as well as the *option to defer* and its consequences in portfolio planning for oil companies. For example, if a project shall be postponed, in order to plan future budget and resources for that (like human resources training), portfolio planning demands both the probability of this option be exercised in the near future and the expected waiting time for this real option be exercised.

Learning means that by exercising one option we generate a positive externality to the other asset (information revelation) so that, depending on the learning outcome from the first exercise, the exercise of the second asset option can become more or less attractive. We'll show that in the paper context the correlation coefficient is a good learning measure to capture this effect. Synergy between two real options means that the joint real option value is higher than the sum of individual real option values. In our case, it means that we can merge the development investment with scale gains, in order to exploit the synergy between the projects, increasing the real option value of joint development.

In addition, the exploratory asset is a *compound real option* because, in case of exercise (by drilling a wildcat well) and in case of success, we get another option, namely the option to develop the discovered oilfield. This compound issue has implications for exercising the exploratory option, as we'll see. Contrasting some previous related literature (e.g., Childs, 1995), the projects here are not mutually exclusive (all exploratory and development projects can be implemented). Instead, both the presence of exploratory project enhances the value of the other exploratory asset (due to the learning) and the presence of development project enhances the value of the other development project (due to synergy). Optionality and correlation drive the value enhancement, as we'll see in this paper.

This paper is organized as follow. In the second section is briefly discussed the previous literature on portfolio of real options. Section 3 presents a simple portfolio case with two assets with compound real options, highlighting the effect of correlation on learning and synergy. Section 4 discusses the option to defer of both exploratory and development options and the implications for portfolio planning. Section 5 put the case from section 3 into a dynamic framework, by considering the option to defer and its interaction with learning and synergy. Section 6 discusses the case of more than two exploratory assets, focusing in learning aspects and presenting a framework named revelation processes, a sequence of conditional expectation distributions. Section 7 set some conclusions and suggestions for future research.

2 – Real Options Literature on Portfolio Theory

The portfolio theory for real assets is much more complex than for financial assets in many aspects. First, the optional nature of real assets: in many cases we have multiple interacting *compound options* embedded in the same asset. Second, *learning* effect due to the information spillover to other assets when exercising the option to invest in one asset correlated with others in the portfolio. Third, the *synergy* effect due to *economy of scale* of joint development options exercise or due to *economy of scope*. Fourth, the *non-divisibility* of projects and practical aspects like the physical resource constrains, demand an adequate intertemporal portfolio resource planning to make feasible the optimal exercise of real options. Fifth, other aspects such as *agency* issues like the incentive for the optimal portfolio management by the firm executives and strategic interactions with other firms, like competition and cooperation opportunities (game-theoretic aspects).

The synergy effect can be viewed as a particular and non-extreme case of super-additive portfolio. The *additive degree* between projects ranges from the extreme *sub-additive* case, i.e., *mutually exclusive* (or competing) projects (e.g., we have two projects with different technologies to produce the same product), to the other extreme *super-additive* case, i.e., one project is a *necessary* *complement* of the other (e.g., a gas field development project and a gas pipeline project linking this area to the market).

The theory of real options interactions *in the same asset* is well developed with the papers of Brennan & Schwartz (1985), Dixit (1989), Trigeorgis (1993), Kulatilaka (1995) and Dixit & Pindyck (2000). However, the portfolio theory for real assets with real options theory lens (i.e., considering options interactions in *different* real assets) is still in development. This paper intends to contribute in some relevant aspects of this theory. In this section is discussed some relevant and rare previous literature.

Brosch (2001) discusses some portfolio aspects for real assets such as the diversification effect, pointing out that the firm cannot create value by diversification (but reduces the risk) and classifying direct qualitative interaction between assets from the extreme of "strictly substitute" to "strictly complementary", passing by the intermediate point of "independent". However, contrasting this paper, instead quantifying the additive degree in function of the dependence (correlation) degree between the assets, he focus the extreme case of "strictly complementary", which is similar to the case of compound options in the same asset.

The other extreme case is the sub-additive portfolio of *mutually exclusive* projects. For the cases without learning, the option to choose one (the maximum value) from n competing assets with correlated market uncertainties, has been analyzed in both financial options literature (e.g., Stulz, 1982; Margrabe, 1978) and in real options literature (e.g., Carr, 1995). In this case, *positive correlation decreases the value of the option* and negative correlation increases the value of the option on a basket of assets. However, when we consider learning and/or synergy, the role of correlation changes, as we'll see in this paper.

Vassolo & Anand & Folta (2004) analyzes a portfolio of exploration assets, with focus on biotechnology applications, showing that the assets can be either sub-additive or super-additive. Their interest are strategic aspects, e.g., one investment opportunity can be super-additive for one firm, but not for other, if this firm has fungible, unused capabilities, to exploit quickly and paying a lower price to exercise this growth option. Their framework highlights the value of technology alliances, viewed as real options, and they make some empirical tests to support their theory.

Luehrman (1998) is a popular article that highlights that strategy is a portfolio of real options, e.g., business strategy is much more a series of options than a series of static cash flows. However, he doesn't quantify the role of correlation over the portfolio value with learning and synergy as here.

Smith (2004) and Smith & Thompson (2004) analyze a portfolio of petroleum exploration assets using real options approach. However, they focus on the specific optimal stopping problem of a drilling sequence of dry holes, not in the issues analyzed in this paper. That papers show that dependence (positive correlation in this case) increases not only the risk, but also the portfolio value.

Childs (1995) is an important contribution to this research topic[1]. His focus of applications is different of this paper mainly because he considers mutually exclusive development projects. However, he discusses learning interactions in previous project phases. As here, he analyzes the case of two assets with compound options: each real asset comprises an exploratory option and a development option[2]. His model doesn't apply to the petroleum exploration & development case because in Childs the development projects are mutually exclusive (only one can be implemented) and because the second asset can be developed without previous exploration (using only the information revealed with the correlated first exploratory project). In contrast with this paper, he considers only European type options and only endogenous uncertainty in each asset (with correlation), but not the exogenous market uncertainty. Childs considers the learning effect between the assets at the exploratory phase, when the exploratory project investments are sequential (Childs also analyses the parallel investment case). By using log-normal distributions (Childs, 1995) or normal distributions (Childs et al, 1998), the learning intensity is given by the square of correlation coefficient ρ^2. Dias (2002, 2005a, 2005b) uses a more general learning measure, the expected percentage of variance reduction η^2 (the *correlation ratio*), but for the Normal type distributions these measures are equal ($\rho^2 = \eta^2$ for Normal distributions).

The optimal order for sequential investments must consider the learning effect of one asset over the other one, so that the first option to be exercised is not always the asset with highest payoff value. As pointed out by Childs (1995, p.50), "*may be preferable to develop a high variance project first, to maximize the uncertainty resolved even if the project has a slightly lower net benefit*". But in addition, for some probabilities distributions, there is asymmetric probabilistic learning effect (affecting the optimal ordering). That is, there is asymmetry in the conditional distributions in terms of learning effect (the relative learning of X | Y can be different of Y | X), so that we can even learn

[1] See also the related paper of Childs & Ott & Triantis (1998).
[2] Childs uses different nomenclature: "development" instead of "exploratory" and "implementation" instead "development". This paper uses the standard oil industry nomenclature. In addition, "exploratory" sounds more appropriate for the uncertainty reduction (learning) that characterizes the first investment phase.

more with the lower variance asset[3]. In this regard, learning between two assets can be either *symmetric* or *asymmetric*, depending on the distributions.

Because learning can be asymmetric, a good must be asymmetric for the general case, even being symmetric in some specific (and important) cases. This paper uses the theory of probabilistic learning measures and the recommended learning measure presented in Dias (2005a, 2005b), the *expected percentage of variance reduction* $\eta^2_{X|Y}$ ($\neq \eta^2_{Y|X}$ in general), also known as *correlation ratio*. For the important cases of X and Y being both Normal distributions (as in Childs et al., 1998) or being both Bernoulli distributions (as here), $\eta^2_{X|Y}$ is symmetric and equal to the square of the popular correlation coefficient (ρ^2), which is convenient for many applications. Appendix A presents a summary of the theory of learning measures and the recommended learning measure used in this paper. This theory, for example, supports the use of the correlation coefficient for learning with Bernoulli distributions.

3 – <u>A Simple Portfolio Case with Two Exploratory Assets: Learning Options and Synergy</u>

In this section is shown the role of probabilistic dependence between real assets in a portfolio is very different of the traditional portfolio theory for financial assets (Markowitz): here there is information revelation by *sequential exercise of learning options*, an active exploitation of dependence, whereas in financial portfolio theory the role of dependence is only for *diversification* purposes. We will show that, for *learning* purposes and in presence of optionality, probabilistic dependence has positive impact on portfolio value for *both* positive and negative correlations, whereas for *diversification* purposes low (even negative) correlation is much better than a higher or positive correlation. In addition, we show that positive correlation is desirable for *synergy* gains, again contrasting financial portfolio theory. The principle of diversification is valid for both financial and real assets portfolio, but in the latter case there are gains with learning and synergy that are not possible for the former.

In this paper are used Bernoulli distributions for the chance factors of exploratory prospects, so that in this case we work with the symmetric learning measure $\rho^2 = \eta^2$ (see Dias, 2005a, 2005b). In the applications exist either positive correlation or negative correlation. A positive correlation application is presented in this paper (with addition to references to a hypothetic case of negative correlation).

[3] Ex.: Let X and Y be discrete uniform distributions with scenarios X ~ U[- 4, - 2, 2, 4] and Y ~ U[4, 16]. Suppose $Y = X^2$ and note that Var[Y] > Var[X]. We learn much more by searching the true value of X (because we get also *full revelation* about the true value of Y) than by searching the true value of the higher variance Y (because X remains stochastic).

A negative correlation application is illustrated in the following example, where two variables (X and Y) are chance factors modeled with Bernoulli distributions and with negative correlation (so that the success in X decreases the success chances of Y). In a P&D problem of a new drug, when searching the cause of a disease, X = {Hypothesis A} and Y = {Hypothesis B} are Bernoulli distributions (so that outcome 1 means that the hypothesis is true and 0 the hypothesis is false). In this kind of application, if we learn that X is equal to 0 (false) in many cases we can say that this information increases the chance of hypothesis B be the real disease cause (i.e., $\Pr[Y = 1 \mid X = 0] > \Pr[Y = 1]$), indicating negative correlation between X and Y. In the extreme case of perfect learning (or full revelation), we have $\rho = -1$ if {Hypothesis B} = {other hypothesis than Hypothesis A}[4]. This example illustrates both that we can learn also with negative correlation and the importance of Bernoulli distributions interactions for other applications than petroleum exploration.

Another simple example that learning is increasing in ρ^2 (rather than ρ) is for Normal distributions, as in Childs et al (1998). If $X \sim N(m_x, \sigma_x^2)$ and $Y \sim N(m_y, \sigma_y^2)$, then it is known that the conditional distribution is also Normal and given by $Y \mid X = x_i \sim N(m_y + \rho\, \sigma_y\, (x_i - m_x) / \sigma_x,\ \sigma_y^2\,(1 - \rho^2))$, i.e, the variance of the conditional (or posterior) distribution, $\sigma_y^2\,(1 - \rho^2)$, is lower as higher is ρ^2, and the maximum learning here (conditional distribution with zero variance, the full revelation case) occurs for both extreme correlation cases, $\rho = +1$ and $\rho = -1$. So, for learning purposes, the correlation signal is not especially important (learning is increasing with ρ^2, not with ρ).

In addition, contrasting the financial portfolio theory, positive correlation between real assets has also positive effects on real assets portfolio value in case of *synergy*[5] between the assets. Negative correlation has negative effect on synergy because decreases the probability of double success (in favor of one success and one failure). In other words, the correlation signal matters for synergy effect (in contrast with learning effect). We illustrate synergy here in terms of economy of scale for the development investment when developing simultaneously two neighboring oilfields (in case of two successes from exploratory drilling), sharing a common infrastructure. In other applications, synergy could be specified in terms of *economy of scope*[6].

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[4] A *necessary* condition for full revelation of Y with the information on X, with X and Y being Bernoulli distributions and with *negative* correlation is *complementary prior success* probabilities, e.g., if $X \sim Be(0.6) \Rightarrow Y \sim Be(0.4)$ for full revelation be feasible. For *positive* correlation, the necessary condition is that the Bernoulli distributions be *exchangeable*.

[5] Synergy between two assets means that the joint two assets value is higher than the sum of individual asset values.

[6] Economies of scope refer to efficiencies primarily associated with demand-side changes, such as increasing or decreasing the scope of marketing and distribution, of different types of products (Wikipedia). When many real options

In this section is considered only the learning and synergy effect as function of correlation in a portfolio of real options. Later we'll include the exogenous market uncertainty and the option to defer. For while, we can imagine that the real option is expiring (it's a now-or-never opportunity). Consider the following example (Dias, 2004 and 2005). An oil company owns a simple exploratory portfolio comprising the rights over a tract with two exploratory prospects. For each prospect, the value of the drilling option exercise is the *expected monetary value* (EMV)[7], given by:

$$\mathbf{EMV_i = -I_W + [CF_i \,.\, NPV_i]}, \qquad i = 1, 2 \qquad \mathbf{(1)}$$

Where I_W is the drilling *investment* in the wildcat well (option exercise price), CF_i is the *chance factor* about the existence of an oilfield for the prospect i, and NPV_i is (conditional to exploratory success) the net present value of the *oilfield development* from the prospect 2 success[8]. The chance factor is the parameter with technical uncertainty with the simplest probability distribution – the Bernoulli distribution, which has two scenarios (1 = success and 0 = failure) and one parameter (p) named success probability. So, we use CF ~ Be(p) to denote this Bernoulli distribution. The expected value of a Bernoulli distribution is the success probability, i.e., E[CF] = p. For simplicity, consider that the *exploratory drilling is instantaneous*, in order to focus on the main paper issues.

Consider initially that the two prospects are *symmetric*, i.e., they have the same parameters and so the same EMV. Assume the numerical values I_W = 30 million \$, E[CF] = p = 30% and NPV = 95 million \$ for both prospects. So, the EMV is negative:

$$EMV_1 = EMV_2 = -30 + [0.3 \times 95] = -1.5 \text{ million } \$$$

Apparently this two real assets portfolio is worthless. Indeed, <u>if</u> the prospects in this portfolio were *independents*, the two-prospects portfolio value would be zero. However, the portfolio value can be strictly positive if the prospects are dependent. Suppose that these two exploratory prospects are in the same geologic play[9], so that the <u>prospects are *dependent*</u> with positive correlation. If these prospects have positive correlation, in case of success in one prospect, the success probability p from the second prospect chance factor (CF_2) *must be* revised upward (to CF_2^+) and in case of failure the

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draw upon a common pool of capabilities (or resources), a firm in several cases can exploit economies of scope with simultaneous option exercise (and/or learning with sequential option exercise strategy).

[7] EMV is used in exploration economics and it is a concept analog to NPV (net present value).

[8] Later in this paper, when considering the option to defer, instead the NPV we'll use the development option value.

[9] The prospects share common geological hypotheses, e.g., existence (or not) of oil migration from the source rock to that area with presence of reservoir rock and synchronism for the sequential geologic events.

probability of success must be revised downward (to CF_2^-). **Figure 1** illustrates this learning process with the information revelation generated by the first option exercise.

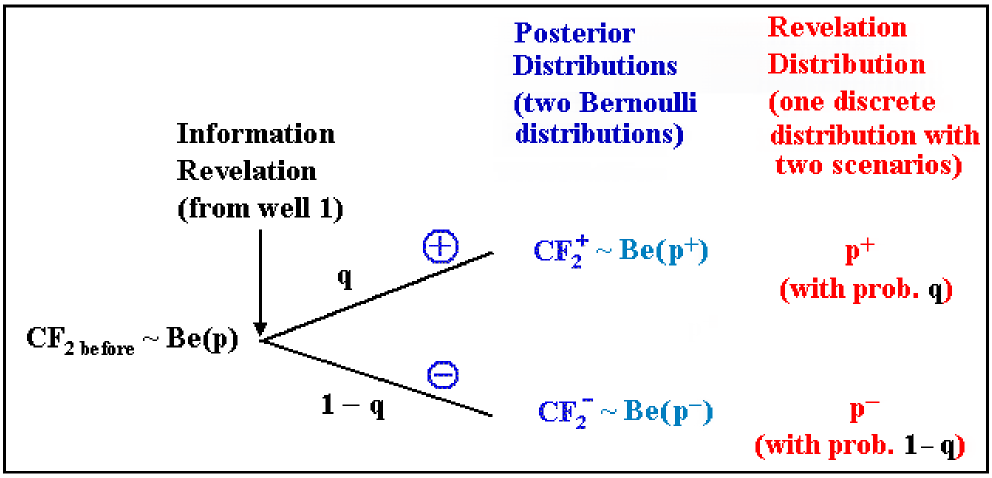

Figure 1 – Effect of the Well 1 Signal on the Chance Factor CF_2

After the signal S_1 (information revelation by drilling the prospect 1), Figure 1 shows two updated scenarios for the 2nd prospect chance factor CF_2: the good news case, $p^+ = E[CF_2 \mid S_1 = CF_1 = 1]$, and the bad news case, $p^- = E[CF_2 \mid S_1 = CF_1 = 0]$, so we have a simple two-scenario discrete distribution of *conditional expectations*, where the conditioning is the information revelation. The distributions of conditional expectations are here named *revelation distributions*, and a set of properties for these distributions is presented in Dias (2002, 2005a, 2005b) and summarized in the Appendix B.

The CF_2 updating process intensity is function of the degree of dependence (correlation) between the prospects and will be quantified soon. The probability of a positive information revelation (q) is the success probability for the well 1. In this symmetrical example, both prospects have the same unconditional success probability (p), so that p = q. In this case these random variables (r.v.) are called *exchangeable*. For notational convenience, considering that the Bernoulli distribution has only one parameter that is also its expected value, instead p and q we use CF_1 (= p) and CF_2 (= q), respectively, for the success probabilities (and so $CF_2^+ = p^+$, etc.).

In this example consider that the dependence degree makes $CF_2^+ = 50\%$ in case of success for the well 1. Probabilistic consistency, given by the *law of iterated expectations*, demands that $CF_2^- = 21.43\ \%$. In case of bad news (i.e., using CF_2^- in the eq. 1), the EMV_2 is even worse than the – 1.5 million obtained with CF_2. But it is an *option* so that the prospect 2 will not be drilled in case of bad news and the value of the prospect in this scenario is zero. However, in case of good news the prospect 2 becomes attractive ($EMV_2^+ = 17.5 > 0$) so that the drilling option is exercised in case of good news. Hence, the portfolio value is:

$$EMV_1 + E[\text{option}(EMV_2)] = -1.5 + [(0.7 \text{ x zero}) + (0.3 \text{ x } 17.5)] = + 3.75 \text{ million \$}$$

A very different value when compared with the case of independent prospects. Note that the positive result is due to both the optional nature of investment drilling *and* the information revelation generated by the first drilling. Thanks to the assets optional nature, the portfolio value is higher as higher is the dependence between the prospects. Hence, the real option value a portfolio of assets with technical uncertainty is an increasing function of the dependence degree between these assets, that here is given by the correlation coefficient ρ (or its square ρ^2).

Consider two Bernoulli random variables, one named the variable of interest with initial chance factor CF_2 and the other named signal with chance factor CF_1. The numbers here is because the prospect 1 will be drilled first, generating signal for the prospect 2 (that learns with prospect 1 option exercise). The updating equations for chance factor in the general case are presented below.

$$\mathbf{CF_2^+ = CF_2 + \sqrt{\frac{1 - CF_1}{CF_1}}\ \sqrt{CF_2\ (1 - CF_2)}\ \rho} \qquad \mathbf{(2)}$$

$$\mathbf{CF_2^- = CF_2 - \sqrt{\frac{CF_1}{1 - CF_1}}\ \sqrt{CF_2\ (1 - CF_2)}\ \rho} \qquad \mathbf{(3)}$$

For the particular case of *exchangeable* random variables, these equations simplifies to:

$$\mathbf{CF_2^+ = CF_2 + (1 - CF_2)\ \rho} \qquad \mathbf{(4)}$$

$$\mathbf{CF_2^- = CF_2 - CF_2\ \rho} \qquad \mathbf{(5)}$$

That is, after an information revelation with learning intensity ρ, the difference between the revealed chance factors $CF_2^+ - CF_2^-$ is just the correlation coefficient ρ, if the Bernoulli distributions are interchangeable. So, in the numerical example the correlation used was 50% – 21.43 % = 28.57 %.

The multivariate distribution literature shows that are necessary limits of consistence for these distributions, i.e., given the marginal distributions, it is not possible *any* dependence intensity. For example, for the example numbers is not possible the case of $\rho = -1$ (we get a negative value if using eq. 4 with this value of ρ). These limits of consistence are named *Fréchet-Hoeffding limits* and for Bernoulli distributions the correlation coefficient has the following limits (Joe, 1997, p.210):

$$\mathbf{Max}\left\{-\sqrt{\frac{CF_2\ CF_1}{(1-CF_2)\,(1-CF_1)}}\ ,\ -\sqrt{\frac{(1-CF_2)\,(1-CF_1)}{CF_2\ CF_1}}\right\}\ \leq\ \rho\ \leq$$

$$\leq\ \sqrt{\frac{\mathrm{Min}\{CF_2\ ,CF_1\}\,(1-\mathrm{Max}\{CF_2\ ,CF_1\})}{\mathrm{Max}\{CF_2\ ,CF_1\}\,(1-\mathrm{Min}\{CF_2\ ,CF_1\})}} \qquad \mathbf{(6)}$$

Now we focus synergy, which is possible in case of double success after drilling both prospects thanks to scale economies with joint development investment. We'll specify synergy in the development investment equation, which is function of the reserve volume. In order to do this, we need work out the NPV function obtained with the development option exercise.

Let the development option exercise payoff for the asset i (NPV_i) be function of the current long-run oil price P. In addition let the NPV_i be also function of both the reserve volume (B, as the number of barrels) and the reserve quality (q, related with the productivity of the reserve and other effects), which are deterministic here. Let us consider a simple *parametric model* named "Business Model"[10] in which the NPV_i obtained with the development option exercise is:

$$\mathbf{NPV_i = q_i\ B_i\ P - I_{Di}} \qquad \mathbf{(7)}$$

Where I_{Di} is the development investment for the oilfield i, conditional to success when exercising the option to drill the exploratory prospect i. The break-even price (P so that NPV = 0) is $P_{be} = I_D/q\ B$, that is the threshold for exercising the development option in this now-or-never case.

The adequate development investment is function of the reserve volume B. Larger volume means larger processing capacity, larger pipeline diameter, larger quantity of development wells, etc. The investment is not proportional to B, but empirical studies show that a linear function is a good approximation for this function, with fixed and variable (with B) factors:

$$\mathbf{I_{Di}(B)\ \ =\ \ k_f +\ k_v\ B_i} \qquad \mathbf{(8)}$$

For the numerical example we'll use the factors $k_f = 180$ and $k_v = 2.5$, with B in millions of barrels and I_{Di} in millions of US$. The index i denotes the asset number (here 1 or 2).

In case of joint investment, we have a synergy gain because it is possible economy of scale by placing a single production unit with higher processing capacity, sharing the same oil and gas pipelines (but with larger diameter). This could suggest applying eq. (8) for the joint reserve volume,

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[10] See a detailed discussion of this and alternative payoff models at www.puc-rio.br/marco.ind/payoff_model.html

$B_1 + B_2$. However, depending on the distance between the oilfields, the flowlines from the wells to the production platform increases so that synergy gain exists but it is not so high. Hence, we adopt a synergy factor γ_{syn}, a number between 0 and 1, representing the synergy intensity: 0 is for no synergy and 1 is for full synergy (here, like a single oilfield with volume of $B_1 + B_2$). The equation below gives the synergy effect over the joint investment of two oilfields.

$$\mathbf{I_{D1+2} \;=\; I_{D1} + I_{D2} - \gamma_{syn}\,[I_{D1} + I_{D2} \;-\, (k_f + \; k_v\,(B_1 + B_2))]} \qquad \mathbf{(9)}$$

When applying this joint investment, in order to calculate the joint development NPV we use an average economic quality q_{1+2}, weighted by the volume of each individual reserve, and the total volume $B_1 + B_2$ in order to calculate the total benefit:

$$\mathbf{NPV_{1+2} \;=\; q_{1+2}\,(B_1 + B_2)\,P \;-\, I_{D1+2} \;=\; (q_1\,B_1 + q_2\,B_2)\,P \;-\, I_{D1+2}} \qquad \mathbf{(10)}$$

For the numerical example, let the (current expectation on long-run) oil price be 30 \$/bbl, the economic quality for both oilfields $q_1 = q_2 = 12\%$, $B_1 = B_2 = 250$ million bbl. So, each isolated NPV values 95 million US\$. Considering a synergy intensity with factor $\gamma_{syn} = 0.5$, in case of joint development we get a $NPV_{1+2} = 280$ million \$ ($> NPV_1 + NPV_2 = 2 \text{ x } 95 = 190$), an expressive gain.

The synergy effect is only possible if we get a double success when exercising the option to drill the exploratory prospects. Denote p_{syn} this probability of double success. This probability (and so the expected synergy gain) is increasing with the correlation coefficient as shown by the following equation (Dias, 2005a or Kocherlakota & Kocherlakota, 1992, p.57):

$$\mathbf{prob_{syn} \;=\; \rho\,\sqrt{CF_1\,(1 - CF_1)\,CF_2\,(1 - CF_2)} \;+\; CF_1\,CF_2} \qquad \mathbf{(11)}$$

Proposition 1: Consider the two exploratory prospects portfolio presented in this section, with chance factors given by Bernoulli distributions with correlation coefficient ρ. The exploratory investment (prospect drilling) is optional. In case of exercise and if the outcome is "success" (oilfield discovery), the firm has the option to develop the oilfield. In case of double success is possible a joint development exercise with investment synergy given by the synergy factor $\gamma_{syn} > 0$. Then:

a) The learning gain from the first exploratory option exercise is increasing (or strictly non-decreasing) with the square correlation coefficient ρ^2.

b) The expected synergy gain with double exploratory option exercise is increasing (or strictly non-decreasing) with the correlation coefficient ρ.

Proof: a) The portfolio value with the first exploratory option exercise is $EMV_1 + E[option(EMV_2)]$. The function $option(EMV_2) = Max[EMV_2, 0]$ is convex and, by the Jensen's inequality, $E[option(EMV_2)] > option(E[EMV_2])$ and this effect is higher as higher is the uncertainty (variance) of $option(EMV_2)$, which here has two scenarios, EMV_2^+ (using CF_2^+) and EMV_2^- (using CF_2^-). Because the distance between CF_2^+ and CF_2^- (and so between EMV_2^+ and EMV_2^-) is increasing with ρ^2 (eqs. 2 and 3) for the same scenario probabilities (CF_1 and $1 - CF_1$, respectively), the variance of $option(EMV_2)$ is increasing with ρ^2. So, the learning/Jensen's inequality effect is increasing with ρ^2.

b) Synergy gain does not depend on correlation, but it occurs only in case of double success. So, the expected synergy gain is increasing with the probability of double success that is increasing with ρ (eq. 11). Hence, the expected synergy gain is increasing with ρ. □

This proposition is illustrated in several numerical computations, with the charts being showed below. Figure 2 isolates the learning and optionality issues in function of the correlation coefficient ρ, that is, does not consider the synergy effect.

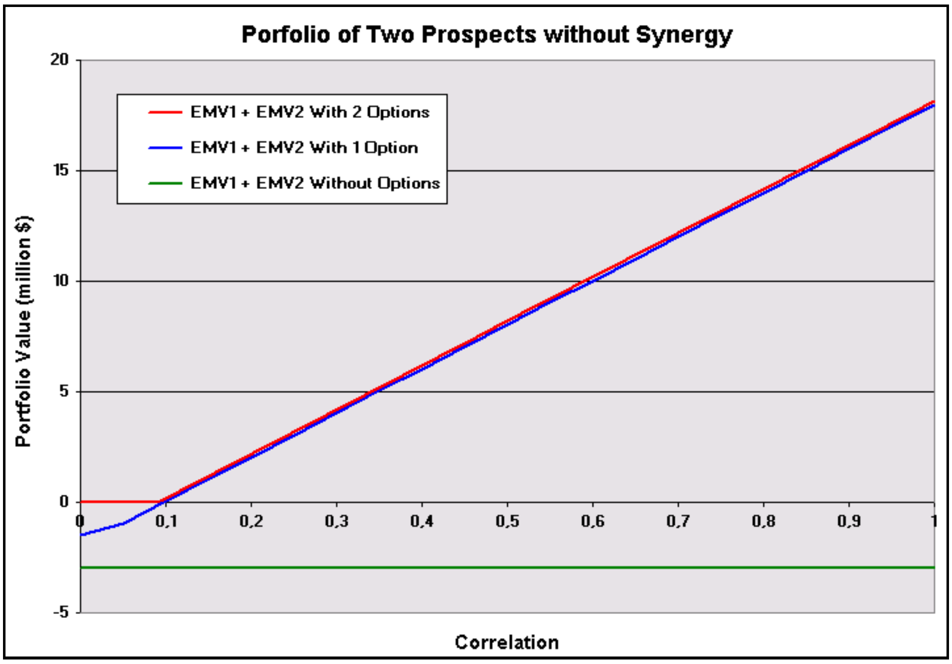

Figure 2 – Two Prospects Portfolio with Positive Correlation and Without Synergy

Without options[11], the portfolio value is negative (- 3 million $) and independent of correlation. With optionality, learning has value and is increasing (strictly non-decreasing) with correlation.

[11] There are cases in petroleum industry where the exploratory drilling is obligatory, due to the "minimal exploratory investment commitment" from the track acquisition bidding process. This obligation can be one or both wells.

The value of the two-exploratory compound options portfolio Π_{1+2} including learning and synergy (in addition to the full optionality) for this expiring opportunity is the sum of EMV_1 with EMV_2 with options, synergy and learning considering that the prospect 1 is drilled first (in case of exercise) and it is given by the following intuitive equation.

$$\Pi_{1+2} = \mathbf{max\{0, - I_W + CF_1\ max[NPV_1, - I_W + CF_2^+\ NPV_{1+2} + (1 - CF_2^+)\ NPV_1] +}$$

$$\mathbf{+ (1 - CF_1)\ max[0, - I_W + CF_2^-\ NPV_2]\}}$$

Figure 3 illustrates this case including the synergy effect. Note that even without options there is an increasing synergy gain with the correlation, which increases the chance of double success.

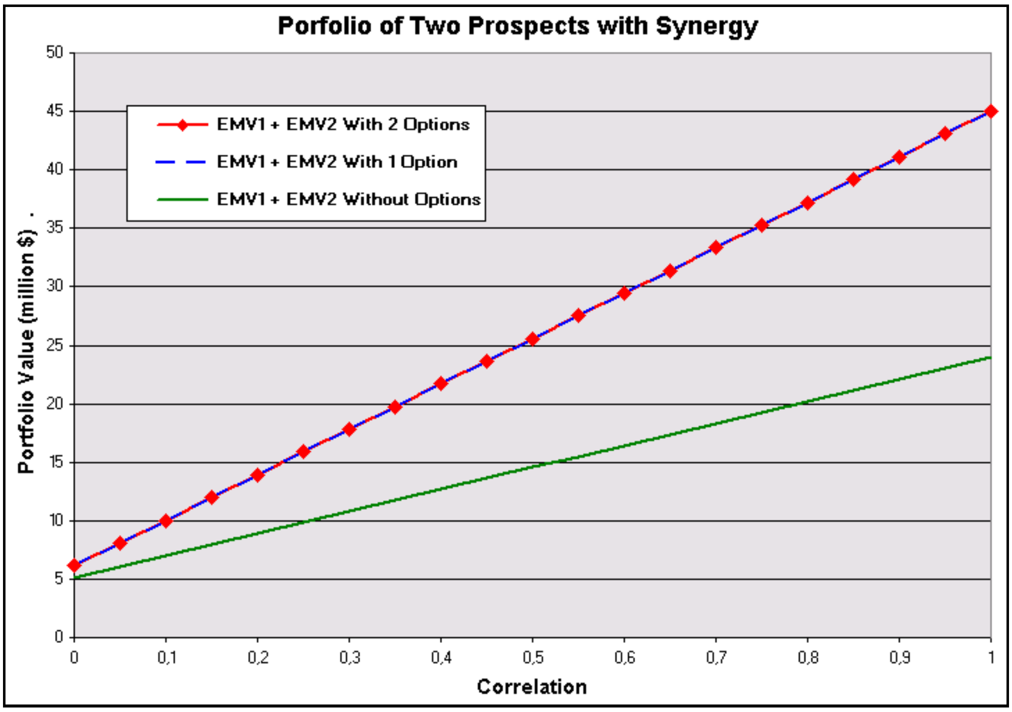

Figure 3 - Two Assets Portfolio with Learning and Synergy: Positive Correlation & Options

In order to complete the theoretical analysis, imagine that is possible negative correlation. Although it is not logic in this petroleum application, we pointed out one real life class of problems where negative correlation is possible/logic. However, as pointed out before, it is not possible the use of any ρ given the (marginal) Bernoulli distributions with parameters CF_1 and CF_2, because ρ must respect the Fréchet-Hoeffding limits (ineq. 6). Figure 4 shows this exploratory example if is allowed negative correlation up to the consistent limits of Fréchet-Hoeffding.

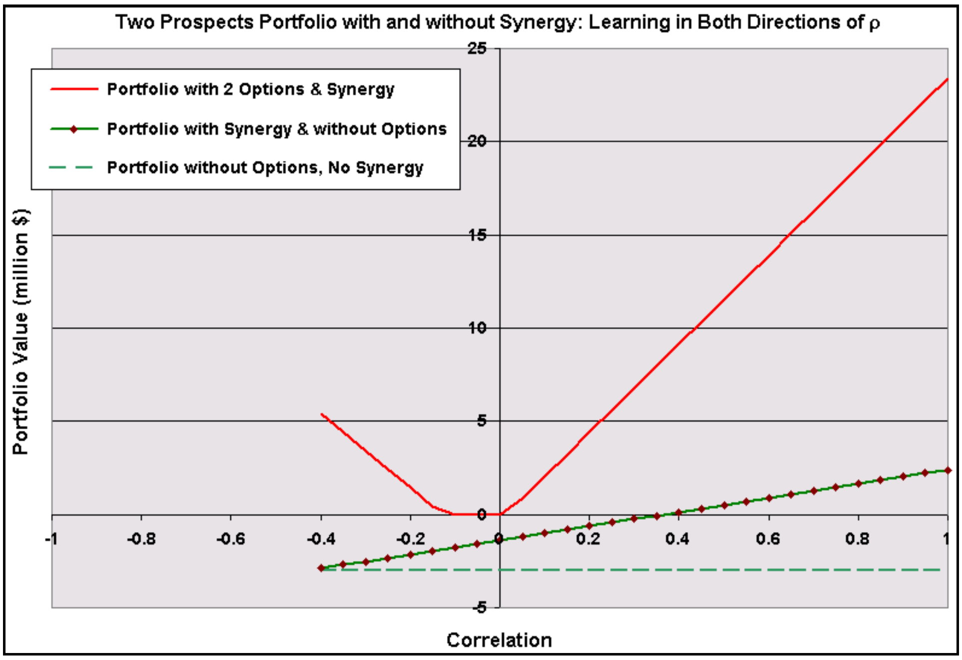

Figure 4 - Two Prospects Portfolio with Positive and Negative Correlations

It is opportune to set the following proposition about the extreme case of learning, the full revelation case, which according our learning theory (see Appendixes A and B) occurs in case of $\rho^2 = 1$, i.e., with either $\rho = + 1$ or $\rho = - 1$.

Proposition 2: Consider the two assets portfolio with chance factors given by Bernoulli distributions with correlation coefficient ρ. A *necessary* condition for maximum learning (full revelation), i.e., for $\rho^2 = 1$, depends on the correlation coefficient signal and is given by:

a) If the correlation coefficient is positive, the necessary condition for maximum learning is the Bernoulli distributions are exchangeable, i.e., with equal success probabilities ($CF_1 = CF_2$).

b) If the correlation coefficient is negative, the necessary condition for maximum learning is the Bernoulli distributions are complementary, i.e., with success probability of one distribution equal to one less the success probability of the other ($CF_1 = 1 - CF_2$).

Proof: By inspection of the inequality for the correlation coefficient, eq. (6). □

Hence, the only case where is allowed all range of correlation coefficient (from –1 to + 1) is when the marginal Bernoulli distributions are simultaneously exchangeable and complementary, i.e., for $CF_1 = CF_2 = 50\%$. We work a numerical example with this modified success probabilities in order to see a chart with the full range of ρ with learning. This is presented in the Figure 5 (case without synergy) and in the Figure 6 (case with synergy).

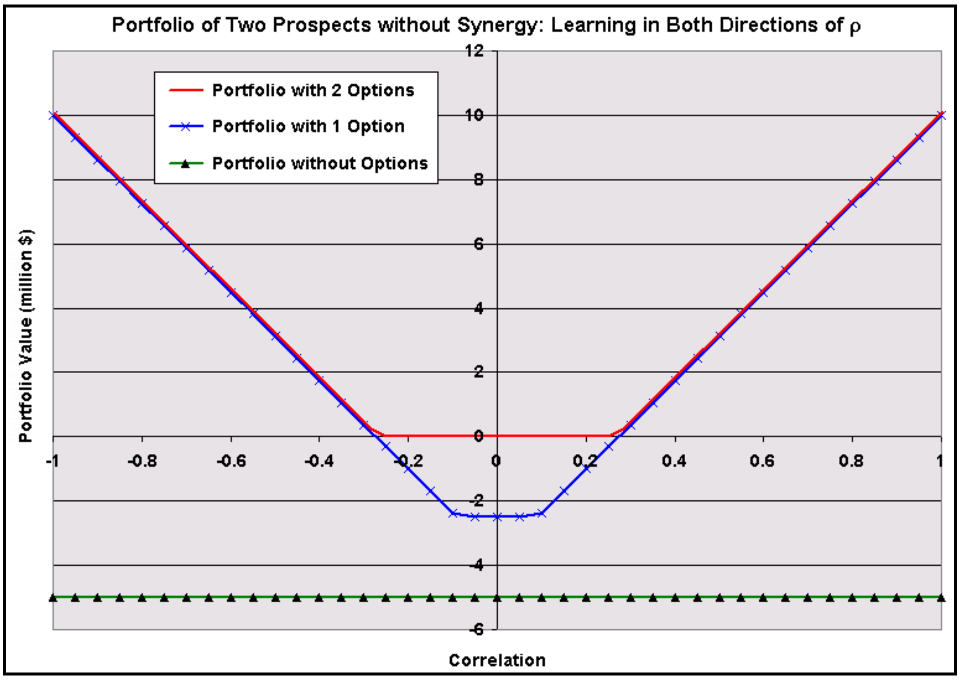

Figure 5 - Two Prospects Portfolio for $CF_1 = CF_2 = 50\%$ without Synergy

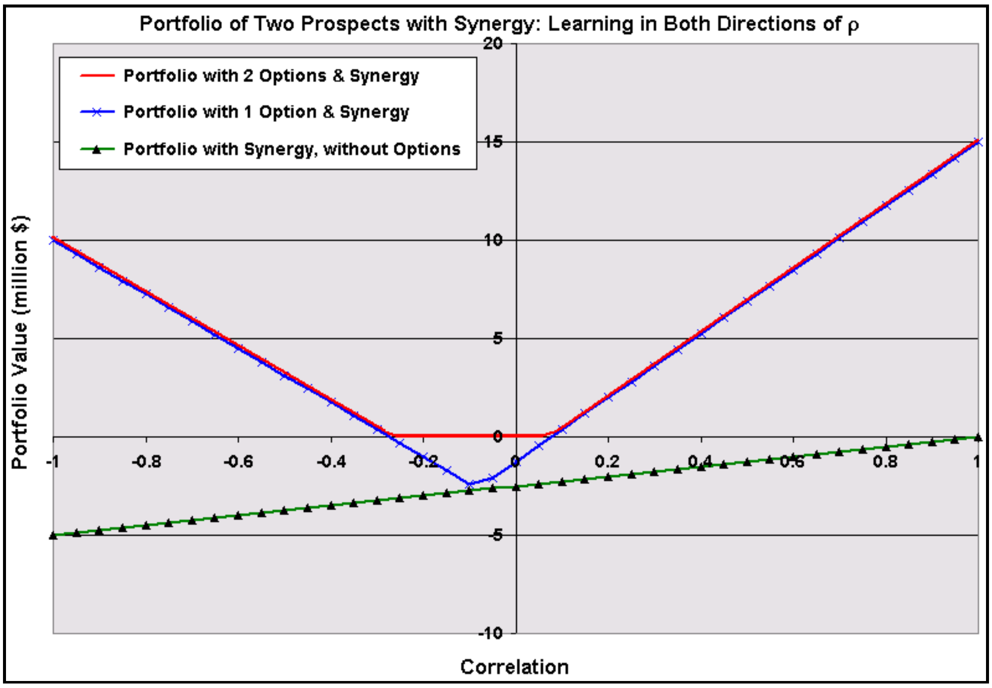

Figure 6 - Two Prospects Portfolio for $CF_1 = CF_2 = 50\%$ with Synergy

Figure 7 presents the same case of Figures 5 and 6, but in the same chart in order to compare all the effects (synergy, learning & optionality).

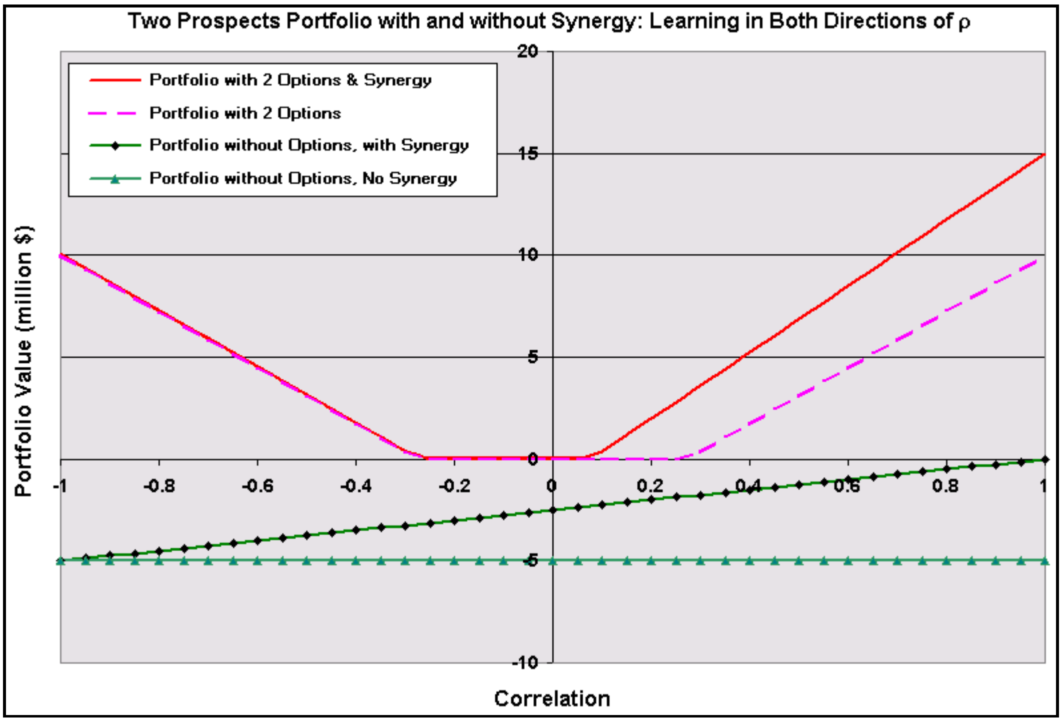

Figure 7 - Two Prospects Portfolio for $CF_1 = CF_2 = 50\%$ with and without Synergy

With these charts is clear the Proposition 1, i.e., that learning is increasing with ρ^2, not with ρ (the correlation signal does not matter), whereas synergy is increasing with ρ (the correlation signal does matter). In the next section is presented the option to defer for the compound exploratory + development petroleum asset, including the discussion of some practical portfolio aspects, while in the section 5 we interact learning, synergy and option to defer compound petroleum options.

4 – Portfolio of Real Assets and the Option to Defer

The option to defer has value in presence of an exogenous market uncertainty that here is represented by the long-run oil price P, which follows a geometric Brownian motion (GBM):

$$\mathbf{dP = \alpha\, P\, dt + \sigma\, P\, dz} \qquad \textbf{(12)}$$

Where α being the drift, σ the volatility and dz is the Wiener increment. Let δ be the oil price (net) convenience yield estimated from the futures market.

Consider that this option is finite, i.e., there is a legal time to expiration regulated by a governmental agency. Following the usual contingent claims steps (build a risk-free portfolio, apply the Itô's Lemma, etc., see, e.g., Dixit & Pindyck, 1994), the value of the development option R(P, t) while alive (not exercised) is governed by the following stochastic partial differential equation (PDE):

$$\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 R}{\partial P^2} + (r - \delta) P \frac{\partial R}{\partial P} - r R + \frac{\partial R}{\partial t} = 0 \quad \textbf{(13)}$$

The optimal exercise conditions are presented as boundary conditions of this PDE, which depends on the development asset characteristics such as q_i, B_i, I_{Di} for development option R_i, being i = 1 or 2, depending on it is the asset number 1 or the asset number 2. In case of joint development (with synergy gain), the option to develop is denoted by R_{1+2} and the joint investment is I_{D1+2}. In this section we focus only one exploratory asset (in the next section we'll need these subscripts to denote the different assets).

Let P* be the threshold (or critical price) for development decision, i.e., at P* is optimal the immediate option exercise developing the oilfield. The boundary conditions, including optimality conditions, are standard in real options literature (e.g., see Dixit &Pindyck, 1994).

- If P = 0, $\mathbf{R(0, t) = 0}$ **(14)**
- If t = T, $\mathbf{R(P, T) = max[NPV(P), 0] = max[q\ B\ P - I_D, 0]}$ **(15)**
- If P = P*, $\mathbf{R(P^*, t) = NPV(P^*) = q\ B\ P^* - I_D}$ **(16)**
- If P = P*, $\dfrac{\partial \mathbf{R(P^*, t)}}{\partial \mathbf{P}} = \mathbf{q\ B}$ **(17)**

This real options problem is solved with numerical methods like finite differences or analytical approximations, which results in both the option value R(P, t) and the optimal decision rule given by the threshold curve P*(t).

Denote the exploratory option value E(P, t; CF) to drill the exploratory prospect as function of the state variables oil price (P) and time (t), highlighting the parameter chance factor CF. Again using the contingent claims method, we obtain a similar PDE, but for the exploratory option E(P, t; CF).

$$\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 E}{\partial P^2} + (r - \delta) P \frac{\partial E}{\partial P} - r E + \frac{\partial E}{\partial t} = 0 \quad \textbf{(18)}$$

As for the development option, there are four boundary conditions for the PDE. But now we shall consider the EMV equation (see eq. 1) when exercising the option to invest by drilling the exploratory prospect. Let P** be the optimal exercise threshold for the exploratory option. Then:

- If P = 0, $\mathbf{E(0, t) = 0}$ **(19)**
- If t = T, $\mathbf{E(P, T) = max[- I_W + CF\ (q\ B\ P - I_D), 0]}$ **(20)**

- If P = P**, $\mathbf{E(P^{**}, t) = -\, I_W + CF\,(q\,B\,P^{**} - I_D)}$ **(21)**

- If P = P**, $\dfrac{\partial \mathbf{E(P^{**}, t)}}{\partial \mathbf{P}} = \mathbf{CF\ q\ B}$ **(22)**

Again this PDE is solved with numerical methods or analytic approximations. Equations 21 and 22 are not obvious because we are saying that when we exercise the exploratory option the development option is already optimal to be immediately exercised, in case of exploratory success. In other words, we are saying that P** ≥ P*. This issue is formalized with the following proposition.

Proposition 3: A *necessary* condition for immediate option exercise of the exploratory prospect is the underlying development option (conditional to exploratory success) be "deep-in-the-money" with positive probability, i.e., in case of exploratory success must be optimal also the immediate exercise of the development option. This implies that is necessary that P** ≥ P*.

Proof: There are at least two ways to see this. First, it is known that a necessary condition to exercise earlier (t < T) an American call option is that the underlying asset pays a positive dividend. This is proved by arbitrage and can be found in good books on option pricing theory. In this case, if the development option R(P**, t) is not "deep-in-the-money" for optimal immediate exercise, then this asset does not generate dividends (cash-flows). Only if R(P**, t) is "deep-in-the-money" is that this option generates "cash flow" because by exercising the development option it transform into an asset that pays dividends (cash flow from the production). Other way to see this is: if R(P**, t) is not "deep-in-the-money", by exercising the exploratory option E(P**, t) we get, in the best scenario (success), the option R and we shall wait because it is not optimal its exercise. In this case, we could be better off if we wait a small interval dt instead exercising the option E, because we delay the investment expense I_W, gaining r I_W dt when compared with the alternative of immediate exercise of option E, without losing any benefit (dividend) from the possibility to have the alive option R. So, it is better to delay the exploratory option exercise if the underlying development option is not "deep-in-the-money" for optimal immediate exercise in case of success. This implies that we must have the necessary condition P** ≥ P*. □

For the theoretical case of an exploratory option with more than one development condition (this case could occur in P&D applications), a necessary condition for the optimal exploratory option exercise is that at least one development option be "deep-in-the-money" for optimal immediate exercise and with positive success probability to occur this development option in case of exploratory option exercise.

In our simple model we consider that the reserve volume and the economic quality are deterministic. A more realistic (but more complex) case could consider them as stochastic so that the exploratory drilling will reveal information about B and q, revising our preliminary estimates for these parameters. In this case, although we exercise our exploratory option expecting that the development option is “deep-in-the-money” in case of success, we can face the situation of exploratory success but bad news in terms of information revelation about B and q. So, its possible to exercise an exploratory option, obtaining success (existence of petroleum), but postponing optimally the development depending on the revealed scenarios of B and q.

A more practical problem regarding the option to defer development is portfolio planning. Oil companies need perform a middle term forecast of resources demand in order to exercise optimally its portfolio of assets at the right time without resources constrains that decreases the portfolio value. For example, rigs and special ships (to launch pipelines and/or flowlines) demand specific contracts where each resource acts in a set of projects. The contracts in general are not project specific. So, if a development project is not “deep-in-the-money”, the oil company needs an idea about the probability of this option to become “deep-in-the-money” until the legal option expiration and, conditional to any exercise later, what is the expected exercise delay for each project. With this information, the oil company can plan new contracts (and the contracts duration), human resources demand in the next years, financing demand in next years, etc.

In order to do this, for each project, the manager shall watch the market evolution and shall be with the threshold curve for optimal immediate investment in her/his hands. The manager will follow this threshold for the development investment decision to be consistent with real options theory.

So, in order to estimate both the probability of option exercise and the expected conditional exercise time, we can perform a Monte Carlo simulation of the stochastic variables (here the oil prices, which follows a GBM) so that when this simulated price reaches the threshold line we consider that the option is exercised. However, in contrast with the option valuation case, we use the real stochastic process for the market (price P) uncertainty, not the risk-neutral simulation. The reason is that the manager will observe the real process, not the risk-neutral one, to checkout the threshold chart for decision purposes. The risk-neutral approach is used in option valuation because we don’t know (or it is very complex to know) the risk-adjusted discount rate for the option. With the risk-neutral simulation, we can valuate the option by using the risk-neutral discount rate to calculate the present value for all simulated paths with exercise. This change of measure is well known (see Girsanov

Theorem in any good mathematical finance book). If hypothetically we know the option discount rates (that changes with the state of the nature), we could use real simulation and these risk-adjusted discount rates. For the probability of option exercise and expected exercise time, doesn't make sense to proceed with a change of measure, as in the case of option valuation.

So, for the probability of option exercise and expected exercise time we shall use real stochastic process simulation associated with the threshold curve obtained from the option valuation process. In this case, contrasting the risk-neutral approach, the GBM drift (α) matters: as higher is the drift, as higher is the probability of option exercise and lower is the conditional expected exercise time.

5 – Combined Effects on Exploratory Assets Portfolio: Learning, Synergy and Option to Defer

In this section is incorporated the option to defer effect, for both the exploratory options and the development options, into a portfolio of two correlated exploratory assets.

The synergy occurs only in case of two successes when exercising the exploratory option. The development option $R_{1+2}(P, t)$ of joint development is given by a PDE and the suitable boundary conditions. The PDE is the same of eq.(13), as well as the first boundary condition (eq. 14). The remaining boundary conditions for the joint development option $R_{1+2}(P, t)$ are listed below, remembering that the joint oilfield development has a joint investment $I_{D1+2}(B_1, B_2, \gamma_{syn})$, given by the eq.(9), which considers the synergy effect, and a exercise payoff given by the eq.(10).

- If $t = T$, $\quad \mathbf{R_{1+2}(P, T) = max(q_{1+2}\,(B_1 + B_2)\,P - I_{D1+2}, 0) = max(NPV_{1+2}(P, T), 0)}$ **(23)**
- If $P = P^*_{1+2}$, $\quad \mathbf{R_{1+2}(P^*_{1+2}, t) = q_{1+2}\,(B_1 + B_2)\,P^*_{1+2} - I_{D1+2} = NPV_{1+2}(P^*_{1+2}, t)}$ **(24)**
- If $P = P^*_{1+2}$, $\quad \dfrac{\partial \mathbf{R_{1+2}(P^*_{1+2}, t)}}{\partial \mathbf{P}} = \mathbf{q_{1+2}\,(B_1 + B_2)}$ **(25)**

Where P^*_{1+2} is the threshold for the optimal joint development option exercise. If the oilfields are equal, with the same individual development threshold P^*, it is easy to see that $P^*_{1+2} \leq P^*$, i.e., synergy speeds up the development. If the oilfields are different, say $P_1^* < P_2^*$, then it is easy to see that in case of $P^*_{1+2} < P_1^* < P_2^*$ we wait while $P < P^*_{1+2}$ and exercise the joint development option otherwise. In addition, if $P_1^* < P^*_{1+2} < P_2^*$, the "wait and see" policy can be optimal even if the current prices $P \in [P_1^*, P^*_{1+2})$ in case of $R_{1+2}(P, t) \geq NPV_1 + R_2(P, t)$. In this case, depending on the problem parameters, is possible to appear *disconnected exercise sets*[12], i.e., interval of P values where is optimal exercise only the oilfield 1 development, followed by a interval of P where waiting is optimal (*intermediate waiting region*) followed by a interval of P where is optimal to exercise the joint oilfield development option R_{1+2}.

Figure 8 illustrates the synergy effect on the two-oilfields portfolio, by comparing the arithmetic sum of two development options ($R_1 + R_2$, without synergy effect) with the joint development option R_{1+2} that considers the synergy effect, for different synergy factors γ_{syn}. The function $R_{1+2}(\gamma_{syn})$ is non-linear and convex with γ_{syn}, although the chart scale doesn't permit see this clearly.

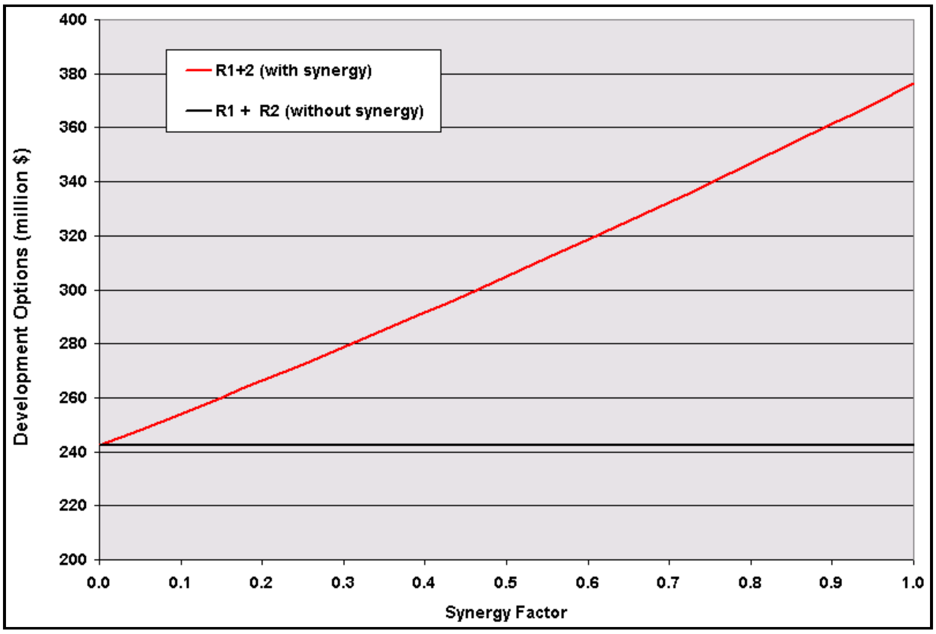

Figure 8 – Option to Develop and Synergy Effect on Two-Oilfields Portfolio

In the above figure the value of the prospects are equal and in this case $R_{1+2} \geq R_1 + R_2$ for all value of the synergy factor. However, if the prospects have *asymmetric values*, it is possible $R_{1+2} < R_1 + R_2$ for low synergy factor values. In this case, the oilfield 2 can "contaminate" the joint development option when single R_1 exercise can be better. Figure 9 shows an example: if the prospect 2 has only the half of reserve volume B_2 of the previous case, for low values of γ_{syn}, we see that $R_{1+2} < R_1 + R_2$.

[12] See the discussion of *disconnected exercise sets*/*intermediate waiting regions* in the context of optimal scale of a single project development in Dias (2004), Dias & Rocha & Teixeira (2003) and Décamps & Mariotti & Villeneuve (2003).

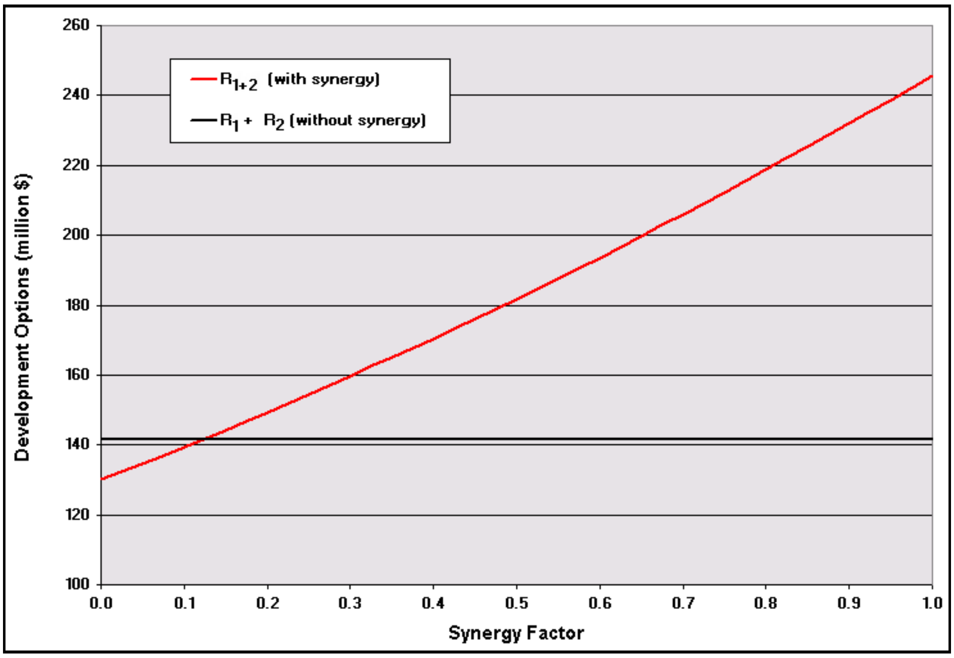

Figure 9 - Option to Develop and Synergy Effect on Asymmetric Two-Oilfields Portfolio

The next backward step for portfolio valuation of these correlated two-compound real options is to analyze the exploratory options and the learning effect, given the synergy opportunity in case of double success presented above and considering the option to wait for better market conditions.

Figure 10 restates the petroleum two-asset portfolio example but including the option to defer. The format is a decision-tree, but it reflects only a specific point in time. At each instant we have the same decision-tree, but with different values for the options and exercise payoffs.

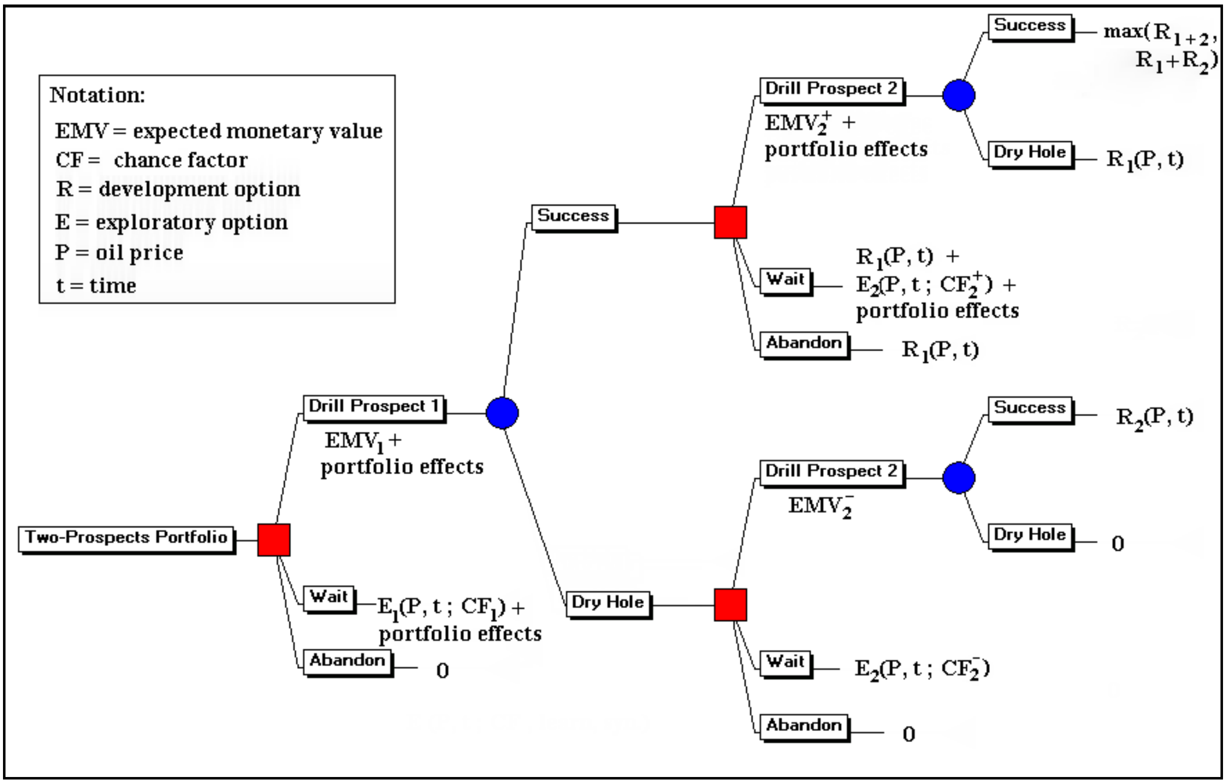

Figure 10 – Two Exploratory Prospects Portfolio Including the Option to Defer

The value of the prospect i exploratory option in presence of a portfolio of exploratory assets is given by the value of this prospect isolated plus the portfolio effect of adding this asset on the portfolio, i.e., learning and synergy effects:

$$\mathbf{E_{i\ portfolio} = E_{i\ isolated} + expected\ portfolio\ effect\ of\ adding\ prospect\ i}$$

In all cases, the PDE for the exploratory option E(P, t) is the same of eq.(18), only the boundary conditions will change in order to consider the different cases showed in Figure 10. The PDE is the same because it depends only on risk-neutral parameters from the stochastic process of P and if the derivative E(P, t) generates or not cash flows (E does not generate cash flow in this case). In addition, the first boundary condition (eq. 19) is the same for all cases. The remaining boundary conditions are presented below for each exploratory option case showed in the Figure 10, capturing the specific learning and synergy effects of each case.

We start backwards in the Figure 10, presenting first the payoff from exploratory option exercise of prospect 2 in case of good news from the first exploratory option exercise, i.e., EMV_2^+ + portfolio effect (Figure 10, top branch). The EMV_2^+ already incorporates the learning effect (learned chance factor is CF_2^+) and the synergy effect is captured under the rubric "portfolio effect". Hence, the expected payoff of this exploratory option exercise, EMV_2^+ + portfolio effect, is:

$$\mathbf{EMV_2^+ + portfolio\ effect\ =\ -\,I_W + CF_2^+\ max\{(R_{1+2} - R_1),\ R_2\}} \qquad \mathbf{(26)}$$

In words, exercising the prospect 2 exploratory option we spend I_W and with probability CF_2^+ we have success obtaining either the joint development option R_{1+2} and giving up the isolated development option R_1, if $R_2 < R_{1+2} - R_1$, or obtaining the isolated development option R_2, if $R_2 > R_{1+2} - R_1$. In case of failure, with probability $1 - CF_2^+$, we don't obtain any additional benefit (in this branch is already guaranteed the portfolio payoff R_1 due to the first success). Equation (26) is the exercise payoff of $E_2(P,\ t;\ CF_2^+)$. As seen before, the necessary condition for earlier exploratory option exercise is the underlying development option (here R_{1+2} and/or R_2) be "deep-in-the-money". So, under this necessary optimal condition, eq.(26) becomes:

$$\mathbf{EMV_2^+ + portfolio\ effect\ =}$$

$$\mathbf{=\ -\,I_W + CF_2^+\ max\{[NPV_{1+2}\ - R_1]}\ \text{ if } R_{1+2} = NPV_{1+2},\ \mathbf{NPV_2}\ \text{ if } R_2 = NPV_2\} \qquad \mathbf{(27)}$$

That is, the eq.(26) conditional to $R_{1+2} = NPV_{1+2}$ and/or $R_2 = NPV_2$, where NPV_{1+2} is given by eq.(10). With the conditionals inside eq.(27), we prevent cases where waiting is optimal even with either $R_{1+2} = NPV_{1+2}$ or $R_2 = NPV_2$. For example, the case of $R_{1+2} = NPV_{1+2}$ but $NPV_{1+2} - R_1 < NPV_2 < R_2$ or the case of $R_2 = NPV_2$ but $R_{1+2} - R_1 > NPV_{1+2} - R_1 > NPV_2$.

Hence, the exploratory option after learning good news, $E_2(P,\ t;\ CF_2^+)$, is given by the PDE, eq.(18) and, in addition to eq.(19), with the following boundary conditions that consider synergy effect:

- If $t = T$,

$$\mathbf{E_2(P,\ T;\ CF_2^+)\ =\ max\{0,\ -\,I_W + CF_2^+\ max[(NPV_{1+2} - max(NPV_1,\ 0)),\ NPV_2]\}} \qquad \mathbf{(28)}$$

- If $P = P_2^{**}$, $R_{1+2} = NPV_{1+2}$ and $NPV_{1+2} > R_1 + R_2$,

$$\mathbf{E_2(P_2^{**},\ t;\ CF_2^+) = -\,I_W + CF_2^+\ (NPV_{1+2}\ - R_1)} \qquad \mathbf{(29a)}$$

- If $P = P_2^{**}$, $R_2 = NPV_2$ and $NPV_2 > R_{1+2} - R_1$,

$$\mathbf{E_2(P_2^{**},\ t;\ CF_2^+) = -\,I_W + CF_2^+\ NPV_2} \qquad \mathbf{(29b)}$$

- If $P = P_2^{**}$, $R_{1+2} = NPV_{1+2}$ and $NPV_{1+2} > R_1 + R_2$,

$$\frac{\partial E_2(P_2^{**}, t; CF_2^+)}{\partial P} = CF_2^+ \left[q_{1+2}(B_1 + B_2) - \frac{\partial R_1(P_2^{**}, t)}{\partial P}\right] \qquad \textbf{(30a)}$$

- If $P = P_2^{**}$, $R_2 = NPV_2$ and $NPV_2 > R_{1+2} - R_1$,

$$\frac{\partial E_2(P_2^{**}, t; CF_2^+)}{\partial P} = CF_2^+ \, q_2 \, B_2 \qquad \textbf{(30b)}$$

Note that there are two mutually exclusive cases for the threshold P_2^{**}: a necessary condition to optimal exercise of E_2 is that in case of success we don't wait optimally. We must optimally exercise either the joint development option R_{1+2} or the isolated oilfield development option R_2. The solution of $E_2(P, t; CF_2^+)$ is obtained by standard numerical methods like finite differences.

The cases that appear after the prospect 1 failure outcome ("dry hole"), the bottom right tree showed in Figure 10, are easier to model because there is no synergy effect anymore, just the prospect 2 exploratory option $E_2(P, t; CF_2^-)$, which learned the bad news by updating the chance factor CF_2 to a lower success probability CF_2^-. So, we can use the eqs.(18)-(22) with CF_2^-. Without the synergy possibility, the value of the development option R_2 is the standard real option given by eqs.(13)-(17).

A little bit more complicated is the case of $E_1(P, t; CF_1)$ with additional benefits of synergy and learning to take into account in order to exercise or not the first exploratory option, showed in the Figure 10 (bottom-left). The intuition says that $E_1(P, t; CF_1)$ + portfolio effect > $E_1(P, t; CF_1)$ isolated. In addition, it is intuitive that we shall exercise E_1 earlier (lower threshold) in presence of portfolio effect than without these effects. The reason is that the exercise payoff is more valuable because there are valuable learning effect and valuable synergy effect (with probability CF_1) in addition to its payoff. Before setting E_1, its exercise payoff equation is presented below.

$$\textbf{EMV}_1 + \textbf{portfolio effect} = -I_W + CF_1 (q_1 B_1 P - I_{D1}) + \textbf{portfolio effect} \qquad \textbf{(31)}$$

Where "portfolio effect" is the effect of first prospect outcome over the remaining portfolio (i.e., prospect 2), which is given by:

$$\textbf{portfolio effect} = CF_1 [E_2(P, t; CF_2^+)] + (1 - CF_1) [E_2(P, t; CF_2^-)] - E_2(P, t; CF_2) \qquad \textbf{(32)}$$

In words, the portfolio effect is the expect value of the prospect 2 exploratory option value *with information revelation* less the prospect 2 exploratory option value *without* this information. So, it is the *net* gain with new information over the option E_2, the learning effect. In addition, it also includes the synergy effect in the $E_2(P, t; CF_2^+)$ term, which is given by eq. (18) and includes the synergy effect at the boundary conditions, given by eqs. (28)-(30b).

Finally, we set the *prospect 1 exploratory option value*, $E_1(P, t; CF_1)$, which considers the portfolio effect over the prospect 2. It is given by the PDE, eq. (18) and, in addition to eq.(19), by the following boundary conditions (OBS: all NPV and options R are functions of P):

- If $t = T$,

$$\mathbf{E_1(P, T; CF_1) = \max[0, \; - I_W + CF_1\, NPV_1 + CF_1\, E_2(P, T; CF_2^+) + (1 - CF_1)\, E_2(P, T; CF_2^-) - - E_2(P, T; CF_2)\,]} \qquad \textbf{(33)}$$

Where: $E_2(P, T; CF_2^+) = \max\{0, - I_W + CF_2^+ \max[(NPV_{1+2} - \max(NPV_1, 0)), NPV_2]\}$;

$E_2(P, T; CF_2^-) = \max\{0, - I_W + CF_2^- \; NPV_2\}$ and $E_2(P, T; CF_2) = \max\{0, - I_W + CF_2 \; NPV_2\}$

- If $P = P_1^{**}$, $R_{1+2} = NPV_{1+2}$ and $NPV_{1+2} > R_1 + R_2$,

$$\mathbf{E_1(P_1^{**}, t; CF_1) = - I_W + CF_1\, [- I_W + CF_2^+\, NPV_{1+2} + (1 - CF_2^+)\, R_1] + + (1 - CF_1)\, E_2(P_1^{**}, t; CF_2^-) - E_2(P_1^{**}, t; CF_2)} \qquad \textbf{(34)}$$

Where $E_2(P, t; CF_2^-)$ and $E_2(P, t; CF_2)$ are given by eqs. (18)-(22), without portfolio effect.

- If $P = P_1^{**}$, $R_1 = NPV_1$ and $NPV_1 > R_{1+2} - R_2$,

$$\mathbf{E_1(P_1^{**}, t; CF_1) = - I_W + CF_1 \; NPV_1 + CF_1\, E_2(P_1^{**}, t; CF_2^+) + (1 - CF_1)\, E_2(P_1^{**}, t; CF_2^-) - - E_2(P_1^{**}, t; CF_2)} \qquad \textbf{(35)}$$

- If $P = P_1^{**}$, $R_{1+2} = NPV_{1+2}$ and $NPV_{1+2} > R_1 + R_2$,

$$\frac{\partial \mathbf{E_1(P_1^{**}, t; CF_1)}}{\partial \mathbf{P}} = \mathbf{CF_1} \left[\mathbf{CF_2^+ \; q_{1+2} \; (B_1 + B_2) + (1 - CF_2^+)} \frac{\partial \mathbf{R_1(P_1^{**}, t)}}{\partial \mathbf{P}} \right] + + \mathbf{(1 - CF_1)} \frac{\partial \mathbf{E_2(P_1^{**}, t; CF_2^-)}}{\partial \mathbf{P}} - \frac{\partial \mathbf{E_2(P_1^{**}, t; CF_2)}}{\partial \mathbf{P}} \qquad \textbf{(36)}$$

- If $P = P_1^{**}$, $R_1 = NPV_1$ and $NPV_1 > R_{1+2} - R_2$,

$$\frac{\partial \mathbf{E_1(P_1^{**}, t; CF_1)}}{\partial \mathbf{P}} = \mathbf{CF_1} \left[\mathbf{q_1 \; B_1} + \frac{\partial \mathbf{E_2(P_1^{**}, t; CF_2^+)}}{\partial \mathbf{P}} \right] + \mathbf{(1 - CF_1)} \frac{\partial \mathbf{E_2(P_1^{**}, t; CF_2^-)}}{\partial \mathbf{P}} -$$

$$-\frac{\partial \mathbf{E_2(P_1^{**}, t; CF_2)}}{\partial \mathbf{P}} \quad \mathbf{(37)}$$

The boundary condition at expiration, eq.(33), is just the choice between the prospect 1 exploratory option exercise considering the portfolio effect, i.e., eqs. (31) and (32), and no exercise (giving up definitely this opportunity). Eq.(34) is the value matching condition at the optimal exercise of E_1, considering that the joint development (which occurs with probability CF_1 CF_2^+) option is "deep-in-the-money". In this case, is also optimal the immediate exercise of E_2 if the outcome from E_1 exercise is "success". Eq.(35) is the value matching condition at the optimal exercise of E_1, considering that the individual development option R_1 is "deep-in-the-money" and is higher than waiting for the joint development option net gain. In case of success outcome with the option E_1 exercise, is optimal the immediate development option R_1 exercise. Eqs.(36) and (37) are the smooth-pasting conditions for the cases presented in eqs.(33) and (34), respectively, i.e., the derivatives $\partial E_1(P_1^{**}, .; .)/ \partial P$. The solution of $E_1(P, t; CF_1)$ is obtained by standard numerical methods like finite differences.

The two-exploratory assets portfolio value, denoted by Π, is the isolated prospect 1 exploratory option without portfolio effects $E_1(P, t; CF_1)_{isolated}$, given by eqs.(18)-(22), plus the prospect 2 exploratory option with portfolio effects CF_1 $[E_2(P, t; CF_2^+)] + (1 - CF_1)$ $[E_2(P, t; CF_2^-)]$, where the option $E_2(P, t; CF_2^+)$ is given by eqs.(18)-(19) and (28)-(30b), which considers learning and synergy, and the option $E_2(P, t; CF_2^-)$ is given by eqs.(18)-(22). That is,

$$\mathbf{\Pi(P, t; CF_1, CF_2, \rho, \gamma_{syn}) = E_1(P, t; CF_1)_{isolated} + CF_1\, [E_2(P, t; CF_2^+)] + (1 - CF_1)\, [E_2(P, t; CF_2^-)]} \quad \mathbf{(38)}$$

With the above equations, the two-assets compound real options portfolio considering learning, synergy and option to delay, is complete.

Figure 11 illustrates the effect of introducing the option to delay in the portfolio, by comparing the case with one year to expiration with the case at expiration (discussed before in the section 3). Note that for low values of $|\rho|$ the synergy effect is more relevant, whereas for high values of $|\rho|$ the learning effect is predominant.

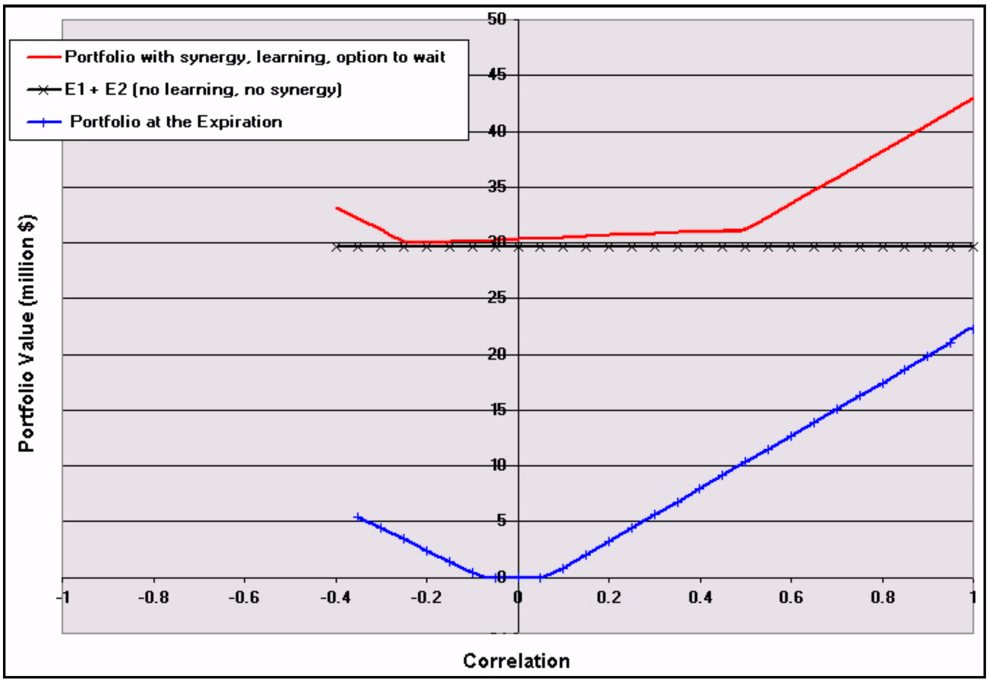

Figure 11 – Portfolio Value with Learning, Synergy and Option to Delay x Correlation

6 – Intertemporal Multi-Asset Exploration Portfolio Valuation & Revelation Processes

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TO BE COMPLETED.

7 – Conclusion

In this paper we study the problem of portfolio of real assets with the real options lens. We examined the effect of correlation on learning and synergy and the optionality role in the portfolio value. The focus was petroleum exploration and development, but the methodology can be applied to other problems such as the R&D portfolio of correlated projects. The learning process study with simple Bernoulli distributions has theoretical interest due to its simplicity, and practical interest because the chance factor is a key variable for exploratory projects. Synergy is considered in the investment, because is common in oil industry that two neighboring oilfields share investment infrastructure.

We saw that the role of correlation in real assets portfolio case is very different of the case of financial assets portfolio. Correlation can create value in the former case thanks to learning plus optionality and synergy.

For learning purposes the correlation signal is not particularly relevant because learning is increasing with the square correlation coefficient. The reason is that for Bernoulli distributions the square correlation coefficient is also the *expected percentage of variance reduction*, a learning measure with many favorable properties.

For synergy purposes, the correlation signal matters and synergy is increasing with the correlation coefficient itself. The reason is that higher correlation increases the chance of double success and so the chance of synergy occurrence.

This paper can be extended in many ways. For example, in addition to chance factor, we could consider learning also in the economic quality of the reserve (q) and in the reserve volume (B). We could also consider some secondary synergy effect over the operational cost, which in this paper model it is captured by the quality q. We could also consider other stochastic processes for oil prices and even stochastic development investment. These and other extensions are left to future work.

APPENDIXES

A) Learning Measures and the Recommended η^2

This section is more detailed described in Dias (2005a, 2005b). Let X be a random variable (r.v.) of interest and S be a signal, also a random variable. The idea is that X learns from S. Formally, consider two r.v. X and S with finite variances, defined in the same probability space (Ω, Σ, $\mathbb{P}$). The following axiom list represents the desirable learning measure properties.

Axioms for Probabilistic Learning Measures: The following axiom list gives the desirable properties for a probabilistic learning measure denoted by M(X | S):

A) M(X | S) shall exist at least for all *non-trivial* r.v. X and S and with *finite* uncertainty;

B) M(X | S) shall be, in general, asymmetric;

C) M(X | S) shall be normalized in the unit interval in order to ease the interpretation, i.e.,

$$\mathbf{0 \leq M(X \mid S) \leq 1} \qquad \textbf{(A1)}$$

D) If X and S are independent $\Rightarrow$ M(X | S) = M(S | X) = 0, because there is no probabilistic learning. In addition,

$$\mathbf{M(X \mid S) = 0 \;\Rightarrow\; \underline{zero\ learning}} \qquad \textbf{(A2)}$$

Where "zero learning" can occur not only for the case of independence. The *learning* concept is defined in the specific measure, but its sense must be *invariable* (eg., the measure of uncertainty shall be the same for all applications);

E) In case of functional dependence, M(X | S) shall be maximum, i.e., for any real function f(.):

$$\mathbf{X = f(S) \;\Rightarrow\; M(X \mid S) = 1} \tag{A3}$$

In addition:

$$\mathbf{M(X \mid S) \;=\; 1 \;\Rightarrow\; \underline{maximum\ learning}} \tag{A4}$$

Where "maximum learning" means that is not possible to learn more about X, and *learning* concept is defined in the specific measure, but its sense must be *invariable*;

F) M(X | S) shall be invariant under linear transformations (changes of scale) of either X or S, i.e., for real constants a and b, $a \neq 0$:

$$\mathbf{M(a\,X + b \mid S) \;=\; M\,(X \mid S)} \tag{A5}$$

$$\mathbf{M(X \mid S) \;=\; M\,(X \mid a\,S + b)} \tag{A6}$$

G) M(X | S) shall be practical in the sense of *easy interpretation* (intuitive) and *easy to quantify/estimate*.

H) M(X | S) shall be additive in the following sense: if the information S can be decomposed into a sum of *independent* factors $S_1 + S_2 + \ldots + S_n$, so that the knowledge of *all* these factors provides *maximum learning*, then the summation of the individual learning measures shall be equal 100%, i.e.:

$$\mathbf{M(X \mid S_1) + M(X \mid S_2) + \ldots + M(X \mid S_n) \;=\; 1} \tag{A7}$$

Definition. **Learning measure η^2**: Consider two r.v. X and S with finite variances, defined in the same probability space $(\Omega, \Sigma, \mathbb{P})$. The expected percentage of variance reduction of X given S is:

$$\boldsymbol{\eta}^2\mathbf{(X \mid S) \;=\; \frac{Var[X] \;-\; E[\,Var[\,X \mid S\,]\,]}{Var[X]}} \tag{A8}$$

The notation η^2 is adopted due to two reasons: (a) it eases the connection with the statistical interpretation of η^2, namely the *correlation ratio*[13], also known by "*eta-squared*" in some statistical books; and (b) in some situations (e.g.: Bernoulli processes) is more intuitive the positive *root* of η^2, i.e., η that is simply the correlation coefficient in the exploratory assets portfolio application. It is easy to prove that:

$$\boldsymbol{\eta^2(X \mid S) \;=\; \frac{Var[\,E[\,X \mid S]\,]}{Var[X]}} \qquad \textbf{(A9)}$$

Proposition (proof: Dias, 1005a, 2005b): Let X and S be two non-trivial r.v.[14] with finite variances, defined in the same probability space $(\Omega, \Sigma, \mathbb{P})$. Consider the learning measure $\eta^2(X \mid S)$ defined above. Then, this measure has the following properties:

(a) The measure $\eta^2(X \mid S)$ always exists;

(b) The measure η^2 is, in general, asymmetric, i.e., $\eta^2(X \mid S) \neq \eta^2(S \mid X)$

(c) The measure η^2 is normalized in unit interval, that is[15],

$$\boldsymbol{0 \leq \eta^2 \leq 1} \qquad \textbf{(A10)}$$

(d) If X and S are independent r.v., then η^2 is zero:

$$\textbf{X and S independent} \;\Rightarrow\; \boldsymbol{\eta^2(X \mid S) \;=\; \eta^2(S \mid X) \;=\; 0} \qquad \textbf{(A11)}$$

In addition, η^2 is zero if and only if the revelation distribution variance is zero:

$$\boldsymbol{\eta^2(X \mid S) \;=\; 0 \;\Leftrightarrow\; Var[R_X(S)] = 0} \qquad \textbf{(A12)}$$

(e) $\eta^2(X \mid S) = 1 \Leftrightarrow$ exists a real function, the r.v. g(S), so that X = g(S);

(f) The measure $\eta^2(X \mid S)$ é invariant under linear transformations of X, i.e., for any real numbers a and b, with $a \neq 0$, we have:

$$\boldsymbol{\eta^2(a\,X + b \mid S) \;=\; \eta^2(X \mid S)} \qquad \textbf{(A13)}$$

(g) The measure $\eta^2(X \mid S)$ is invariant under linear and nonlinear transformation of S if the transformation g(S) is a 1-1 function (invertible function).

[13] The famous statistician Karl Pearson introduced the correlation ratio in 1903. Kolmogorov, 1933, p.60, linked this concept with the conditional expectations concept.

[14] Non-trivial means strictly positive variances. Proposition 1 is valid almost surely (with probability 1).

[15] We could also highlight that η^2 is a *truly* measure, because $\eta^2 \geq 0$.

$$\boldsymbol{\eta^2(X \mid g(S)) = \eta^2(X \mid S)}, \qquad \text{g(s) is invertible} \qquad \textbf{(A14)}$$

In general, for any g(S) measurable by the sigma-algebra generated by S, then the inequality below holds:

$$\boldsymbol{\eta^2(X \mid g(S)) \le \eta^2(X \mid S)}\text{, with equality if g(s) is invertible} \qquad \textbf{(A15)}$$

(h) If the r.v. Z_1, Z_2, … are <u>independent and identically distributed</u> (iid) and if $S = Z_1 + Z_2 + \ldots + Z_j$ and $X = Z_1 + Z_2 + \ldots + Z_{j+k}$ for any non-negative integers j and k, with $j + k > 0$, the proposed measure $\eta^2(X \mid S)$ is given directly by:

$$\boldsymbol{\eta^2(X \mid S) = \frac{j}{j+k}} \qquad \textbf{(A16)}$$

(i) Let the signals S_1, S_2, … , S_n, be independent random variables. We want to learn about X, a random variable with Var[X] > 0. Assume as finite all the relevant expectations and variances. Let $X = f(S_1) + g(S_2) + \ldots + h(S_n)$, where f, g, . . . , h, are any real valued functions. Then:

$$\boldsymbol{\eta^2(X \mid S_1) + \eta^2(X \mid S_2) + \ldots + \eta^2(X \mid S_n) = 1} \qquad \textbf{(A17)}$$

<u>Proposition</u> (Learning measure η^2): The proposed learning measure η^2 obeys the entire axiom list.

<u>B) Revelation Distributions</u>

The proposed learning measure η^2 is related with the concept of *revelation distribution* presented in Dias (2002). Revelation distribution is a distribution of conditional expectations where the conditioning is the information (signal) revealed by the exercise of a learning option. The term "revelation" emphasizes a <u>process towards the true value of variable with technical uncertainty</u>, and it has been used in related literature and before in the classic economics of information literature. This term suggest a learning process to find out the true state of nature.

Revelation distributions are distributions of conditional expectations. Denote the r.v. associated with the revelation distribution by $\mathbf{R_X(S) = E[\,X \mid S\,]}$, where X is the *variable of interest* with technical uncertainty (e.g., chance factor of an oil prospect; reserve volume of a new oilfield) and S is the *signal* (e.g., the drilling outcome from a correlated oil prospect; the information generated by an appraisal well in a new oilfield).

<u>Definition</u>. *Revelation process* is the sequence of r.v. $\{R_{X,1}, R_{X,2}, R_{X,3}, \ldots\}$ generated by a sequence of signals S_1, S_2, S_3, … about an interest variable X, which its main characteristic is the <u>expected reduction of uncertainty</u> provided by these signals. Revelation process is a *probabilistic learning*

process. In the mathematical literature is sometimes referenced as "*accumulating data about a r.v.*" or as *Doob-type martingale*.

Revelation processes are uniformly integrable, meaning that it converges with vanishing risk to an integrable random variable and in particular to the full revelation of X (see Dias, 2005a, 2005b).

Definition. *Full revelation* of X is the revelation of a scenario c so that $\Pr(X = c) = 1$, where c is a constant belonging to p(x) support. In general, if the available information is given by the sub-sigma-algebra Ψ, full revelation of X means that X is Ψ–measurable and, hence, we can write $E[X \mid \Psi] = X$ almost surely (a.s.). Intuitively, it means that there is *perfect information* about the true state of nature for the variable X.

Proposition (*Revelation Distributions*): Let the r.v. X and S two-times integrable (i.e, finite mean and finite variance) defined in the probability space $(\Omega, \Sigma, \mathbb{P})$. The interest variable X has prior distribution p(x). The signal S generates the sigma-algebra Ψ, a sub-sigma-algebra of Σ, i.e., $\Psi \subseteq \Sigma$. Let $p(R_X)$ be the probability density of $R_X = E[X \mid S]$, i.e., the revelation distribution of X given S. Then, the revelation distribution is almost defined[16] by the following properties:

(a) In the limit case of full revelation, the variance of any posterior distribution is zero and the revelation distribution $p(R_X)$ is equal to prior distribution p(x).

(b) The revelation distribution mean is equal to the prior mean of X, i. e.:

$$\mathbf{E[R_X] = E[X]} \qquad \textbf{(B1)}$$

(c) The revelation distribution variance is simply the *expected variance reduction* of X caused by the signal S, i.e., the prior variance less the expected posterior variance:

$$\mathbf{Var[R_X] \;=\; Var[X] \;-\; E[\,Var[X \mid S\,]\,]} \qquad \textbf{(B2)}$$

It can also written using the learning measure η^2:

$$\mathbf{Var[R_X] \;=\; \eta^2\, Var[X]} \qquad \textbf{(B3)}$$

(d) Consider a sequential exercise of learning options generating the signals $S_1, S_2, S_3, \ldots$ and the r.v. $\{R_{X,n}\} = \{E[X \mid S_1, S_2, \ldots S_n]\}$, n = 1, 2, … Then, the *revelation process* $\{R_{X,1}, R_{X,2}, R_{X,3}, \ldots\}$ is a martingale.

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[16] Definition: almost defined distribution is a distribution that we know at least the *mean*, the *varianc*e and that belongs to a *sequential process* of distributions with *known initial distribution* and *convergent to a known distribution*.

Revelation distribution does not require risk-adjustment to use the risk-neutral approach (as in case of market uncertainty distributions) because technical uncertainty does not demand risk-premium from diversified investors. So, *revelation distributions are naturally risk-neutral*. With the above properties, Dias (2002) combines revelation distributions with risk-neutral (or adjusted) stochastic process into a Monte Carlo framework in order to solve the real options problem on investment in information in an oilfield with technical uncertainties before the development decision.

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