

Symmetrical Duopoly under Uncertainty

The Huisman & Kort Model

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1) Introduction

The Symmetrical Duopoly under Uncertainty Model was the first known "option games" model (Smets, working paper in 1991 and doctoral dissertation in 1993), and was analyzed in chapter 9 of Dixit & Pindyck's textbook (1994). In addition to the historical relevance, this is model has a great theoretical importance - foundations of *stochastic* timing-games of preemption in continuous-time - and also practical significance for duopoly when there is no competitive advantage for one firm.

This section follows the in-depth analysis of this symmetrical duopoly performed by **Huisman & Kort**: "*Effects of Strategic Interactions on the Option Value of Waiting*", CentER Discussion Paper 9992, Tilburg University, September 1999, 41 pp. The paper is (or was) [available to download from Tilburg University website](#).

This paper is a thorough (definitive?) analysis for this very basic symmetrical option game. They extended the previous literature (Smets, Dixit & Pindyck) in many ways, specially by allowing *mixed-strategies* equilibrium and analyzing the possibility of non-binding *collusion equilibria*. In addition, they presented many propositions that establish the conditions for the occurrence of different equilibria, in a rigorous approach.

With very few modifications, this paper is also the chapter 7 from the very good textbook of Huisman on option-games in continuous-time: "*Technology Investment: A Game Theoretic Real Options Approach*", Kluwer Academic Pub., Boston, 2001.

I suggest the reading of the Huisman & Kort paper (or the chapter 7 of Huisman's book), but the explanation here is self-contained - although summarized in some topics - and it is complemented with many charts and an Excel spreadsheet (which checks out all proposition, calculated values and tresholds,

presents charts, tell the optimal equilibrium, etc.). I provide also additional explanations in some points of this model. The previous reading of chapter 9 (last section) of Dixit & Pindyck is also very useful. The explanation here presents only some minor modifications in notation of Huisman & Kort. However, I include some issues not discussed in the paper or book, e.g., the *two ways to calculate the value of the leader*.

This duopoly model is symmetrical in the sense that the players are homogeneous firms (equal firms), so that this suggests symmetrical strategies.

I analyze the *asymmetrical* duopoly case in [other webpage \(model of Joaquin & Buttler\)](#), when firms have different costs and so there is a competitive advantage for the low-cost firm.

This webpage is divided into 7 topics. First, this introduction. Second, the model description, the assumptions, the monopoly case, and the follower value and threshold. Third, the leader value and threshold, including the two equivalent ways to calculate the leader value, the value of simultaneous exercise, and the original paper example with some charts to illustrate the duopoly discussion. Fourth, the discussion on mixed strategies equilibria and the probability of mistake - simultaneous exercise when the optimal is the leader/follower schema with only one investment exercise. Fifth, the value of collusion and the conditions for collusion without binding contracts. Sixth, the analysis of equilibrium scenarios, pointing out the more probable equilibrium scenario - collusion or preemption. Seventh, this webpage concludes with the Excel spreadsheet, which is a component from the software pack "Options-Game Suite".

Warning: this page uses the **Html 4.0** code for Greek letters. Use a modern browser for better understanding (Internet Explorer 5.5+ or Netscape 6+). If you are looking "σ" as the Greek letter "sigma" (not as "?", "s" or other thing), OK, your browser understand this Html version.

For the readers with older browsers, the alternative is to download the **pdf version** of this page ([click here to download duopoly2.pdf](#)), but the charts quality are not so good as this webpage.

2) Model Assumptions, Monopoly Value, Duopoly and Follower

The first difference of Huisman & Kort model - when compared with Smets / Dixit & Pindyck - is that the two firms are already active in the market producing one unit each, and they are considering the exercise of (a definitive) one option to expand the production. This means that the "entry" (read expansion) of a firm affects the current profit flow of the other firm - reducing the profit flow because this model consider *negative externalities*.

If firm 1 exercise the option by investing I and firm 2 doesn't invest, firm 1 is called the *leader* and firm 2 is called the *follower*.

The firms face a (inverse) demand curve expressed by the *profit flow* $P(t)$ for firm i given by:

$$P(t) = Y(t) D(N_i, N_j)$$

Where $Y(t)$ is a (multiplicative) stochastic demand chock and $D(N_i, N_j)$ is a deterministic demand parameter for firm i , which depends of the status of firms i and j . The possible values of $D(N_i, N_j)$ are:

- $D(0, 0)$ means that both firms not invested yet (but there is a profit flow $Y D(0, 0)$ because the

- firms are already active in the market);
- $D(1, 0)$ means that firm i invested and is the "leader" because the firm j not invested yet;
 - $D(0, 1)$ means that firm i is the "follower" because only the other firm (j) invested becoming the leader; and
 - $D(1, 1)$ means that both firms invested in the market (simultaneous investment).

These factors $D(N_i, N_j)$ are adjusted to the production level. The operational cost is zero or it is also included in D , so that P is interpreted as profit flow instead price.

For example, the profit flow of the leader is $Y \cdot D(1, 0)$, and this profit flow is higher than $Y \cdot D(0, 0)$. That is, the higher production level when investing and assuming the leader role, is already included in the value $D(1, 0)$. The change of status from $D(0, 0)$ to $D(1, 0)$ demands the sunk investment of I . In addition, due to the symmetry of the problem, when one firm is profiting $Y \cdot D(1, 0)$, the other firm is profiting $Y \cdot D(0, 1)$, etc.

For the called "*new market model*" - the original case of Smets and in Dixit & Pindyck, we have $\mathbf{D(0, 0) = D(0, 1) = 0}$.

The stochastic demand chock $Y(t)$ follows a geometric Brownian motion, given by:

$$dY = \alpha Y dt + \sigma Y dz$$

The model assumes that firms are risk-neutral. However, it is ease and direct to extend to risk-averse firms in contingent claims approach by supposing that the drift of the stochastic process α is a risk-neutral drift. In practical terms, just let $\alpha = r - \delta$ in the relevant equations presented below, with the dividend yield $\delta > 0$. The value of δ can be estimated as the difference between the risk-adjusted discount rate and the real drift of the process.

The deterministic demand parameters have the additional constrain of **negative externalities** give by the inequality:

$$\mathbf{D(1, 0) > D(1, 1) > D(0, 0) > D(0, 1)}$$

Other assumption in Huisman & Kort's model is the *first mover advantage*, given by the inequality:

$$\mathbf{D(1, 0) - D(0, 0) > D(1, 1) - D(0, 1)}$$

This inequality says that the gain when becoming leader is higher than the gain when becoming follower (for the same Y , and considering the same investment I).

Before any options exercise (when Y is below the leader threshold Y_L), the value of each firm is the current cash-flow profit in perpetuity, $Y \cdot D(0, 0) / (r - \alpha)$, plus the option to exercise the investment option (as leader or as follower, with 50% each - as we will see later) net of the competitive losses due to the negative externalities of the rival entry.

First, let us recall the monopoly case (only one firm, no menace of preemption) for comparison reasons and because in the duopoly the leader experiments a monopoly phase before the follower entry. After the monopoly result will be presented the duopoly case and the equilibrium strategies.

Monopoly

In the monopoly case, we have the traditional real options problem, and the value of the firm $V(Y)$ is the current profit plus the option to expand the production. The differential equation is given below, where subscripts means derivatives:

$$0.5 \sigma^2 Y^2 V_{YY} + \alpha Y V_Y - r V + Y D(0, 0) = 0$$

The blue part of the equation is the homogeneous part whereas the red term is the non-homogeneous term (the cash-flow term) of the differential equation. For the case of risk-averse firm, using contingent claims approach, just replace the real drift α by the risk-neutral drift $(r - \delta)$ in the differential equation above.

The whole solution comprises the general solution for the homogeneous part (option) plus the particular solution from the non-homogeneous part (cash-flow), if the stochastic variable Y is below the optimal exercise threshold for the monopolist Y_M . In case of Y equal or higher the threshold Y_M , the value is the monopolist profit in perpetuity net of the investment. See the equation below.

$$V(Y) = \begin{cases} A_1 Y^{\beta_1} + \frac{Y D(0, 0)}{r - \alpha} & \text{if } Y < Y_M \\ \frac{Y D(1, 0)}{r - \alpha} - I & \text{if } Y \geq Y_M \end{cases}$$

Where β_1 is the positive (and > 1) root of the quadratic equation $0.5 \sigma^2 \beta^2 + (\alpha - 0.5 \sigma^2) \beta - r = 0$. See Dixit & Pindyck for a nice discussion on these quadratic equations for the homogeneous part of the differential equation. Again, for the case of risk-averse firm, using contingent claims approach, just replace the real drift α by the risk-neutral drift $(r - \delta)$, so that the denominator $(r - \alpha)$ becomes equal to δ in the equations above.

The values of the constant A_1 and the monopolist threshold Y_M are obtained with the value-matching and smooth-pasting conditions at the point where $Y = Y_M$ (see for example Dixit & Pindyck for these typical boundary conditions). These values are:

$$A_1 = \frac{Y_M^{1 - \beta_1}}{\beta_1} \frac{D(1, 0) - D(0, 0)}{(r - \alpha)}$$

$$Y_M = \frac{\beta_1}{\beta_1 - 1} \frac{(r - \alpha) I}{D(1, 0) - D(0, 0)}$$

The NPV from the option exercise at the threshold Y_M is ease to prove that is $I / (\beta_1 - 1)$, which is strictly positive.

The Follower Value and Threshold

As standard in timing games, the solution is performed *backwards*. This means that *first* we need to estimate the value of the *follower* (given that the leader entered before), and then the leader value given that the leader knows the optimal follower enter can happen in the future. Here is considered that any firm can become the leader (the roles are not exogenously assigned).

Note that the follower has a perpetual option to invest I and obtaining the follower cash flow in perpetuity.

By following the usual steps, the follower value is given by:

$$F(Y) = \begin{cases} B_1 Y^{\beta_1} + \frac{Y D(0, 1)}{r - \alpha} & \text{if } Y < Y_F \\ \frac{Y D(1, 1)}{r - \alpha} - I & \text{if } Y \geq Y_F \end{cases}$$

The values of the constant B_1 and the follower threshold Y_F are as usual determined by the boundary conditions value-matching and smooth-pasting. These values are:

$$B_1 = \frac{Y_F^{1-\beta_1}}{\beta_1} \frac{D(1, 1) - D(0, 1)}{(r - \alpha)}$$

$$Y_F = \frac{\beta_1}{\beta_1 - 1} \frac{(r - \alpha) I}{D(1, 1) - D(0, 1)}$$

Note that the constant B_1 is *positive* reflecting the positive value of the option to invest as the follower. The function $F(Y)$ is *convex* as consequence of the positive value of this constant.

The follower strategy is to invest at the time $T^* = T_F = \inf(t \mid Y(t) \geq Y_F)$.

3) Leader Value and Threshold, and Simultaneous Exercise Value

The leader value is a little bit more interesting. There are at least two ways to calculate this value. The first is to follow as in the paper and as in Dixit & Pindyck. The other way, not presented in the paper or in the Dixit & Pindyck, is perhaps simpler. Both are presented here (finding equal results).

The first approach divides the problem into two expectations. The monopolist phase of the leader (when

the follower not entered yet), with high profits, and the duopoly phase, when the follower entered and the leader profit flow drops with the additional follower supply to the market. Let T^* be the first time that the path of the stochastic process of Y hits the threshold Y_F .

The figure below shows the leader value in this way:

$$L(Y) = \underbrace{\mathbf{E} \left[\int_0^{T^*} e^{-rt} Y(t) D(1, 0) dt \right]}_{\text{Expected Profit in Monopoly Phase}} + \underbrace{\mathbf{E} \left[\int_{T^*}^{+\infty} e^{-rt} Y(t) D(1, 1) dt \right]}_{\text{Expected Profit in Duopoly Phase}} - \mathbf{I}$$

We can write this equation as:

$$L(Y) = D(1, 0) \mathbf{E} \left[\int_0^{T^*} e^{-rt} Y(t) dt \right] + \mathbf{E} \left[e^{-rT^*} \right] \frac{Y_F D(1, 1)}{r - \alpha} - \mathbf{I}$$

The calculus of these expectations with stochastic time T^* is similar to the case presented in other page. These expectations value:

$$\mathbf{E} \left[e^{-rT^*} \right] = \left(\frac{Y}{Y_F} \right)^{\beta_1}$$

$$\mathbf{E} \left[\int_0^{T^*} e^{-rt} Y(t) dt \right] = \frac{Y}{r - \alpha} \left[1 - \left(\frac{Y}{Y_F} \right)^{\beta_1 - 1} \right]$$

[Click here for the proof of the first expectation](#) and for the proof of the [second expectation click here](#).

In this way, by substituting the results into the $L(Y)$ expression we get the value of the leader as function of the leader optimal entry threshold, the parameters from the stochastic process of the demand, and the deterministic demand parameters in monopolist and duopolist phases, $D(1, 0)$ and $D(1, 1)$, respectively.

The alternative (easier!) way is to consider the differential equation of the value of the leader *during the monopolist phase*. This value needs match the value of simultaneous investment (follower value) at the boundary point $Y = Y_F$. So, it is ease to see that the differential equation of the leader value *during the monopolist phase* is:

$$0.5 \sigma^2 Y^2 V_{YY} + \alpha Y V_Y - r V + Y D(1, 0) = 0$$

The solution is the general solution for the homogeneous part (blue part of the differential equation) plus the particular solution for the non-homogeneous term (cash-flow). The solution is:

$$V(Y) = B_2 Y^{\beta_1} + \frac{Y D(1, 0)}{r - \alpha}$$

The constant B_2 is the parameter that remains to be calculated, requiring only one boundary condition for that. The biggest difference compared with the constant B_1 from the value of the follower, is that the constant B_2 is negative, reflecting in the (expected) leader value, the losses due to the probable follower investment exercise. This is mathematically showed below.

The relevant boundary condition is the *value-matching* at the point that the follower entry (at Y_F). The smooth-pasting condition is not applicable here because this point is not an optimal control of the leader, it is derived of one optimization problem but from the *other player*. This boundary condition is:

$$V(Y_F) = Y_F D(1, 1) / (r - \alpha)$$

The leader value *during the monopolist phase* is equal to the simultaneous investment value at Y_F . Equaling the two last equations, we get:

$$B_2 Y_F^{\beta_1} + \frac{Y_F D(1, 0)}{r - \alpha} = \frac{Y_F D(1, 1)}{r - \alpha}$$

The value of the constant is easily obtained:

$$B_2 = \frac{Y_F [D(1, 1) - D(1, 0)]}{Y_F^{\beta_1} (r - \alpha)}$$

Note that the constant value is negative because $D(1, 1) < D(1, 0)$. This means that the effect of the follower entering is to decrease the leader value, as expected by the economic intuition in this duopoly. The negative value of the constant means that the leader value function is concave. The leader value in the monopoly phase $V(Y)$ is obtained by substituting this constant into the leader equation.

Let define the value of *becoming leader* as $L = V - I$. In reality, both Dixit & Pindyck and Huisman & Kort, denote L the leader value (but not Grenadier, 2000) probably because it is direct the comparison of L with F in the choice of the optimal strategy (and to check the incentives to deviate of strategy).

If $Y < Y_F$, the **leader value** defined as $L = V - I$, is given by:

$$L(Y) = \frac{Y D(1, 0)}{r - \alpha} + \left(\frac{Y}{Y_F}\right)^{\beta_1} \frac{Y_F [D(1, 1) - D(1, 0)]}{r - \alpha} - I$$

The reader easily can check this value is exactly the same obtained by the first method (and we didn't calculate any expectation with random time).

If $Y \geq Y_F$, the value of becoming leader is equal the value of becoming follower, which is equal the value of simultaneous investment. That is,

$$L(Y) = \frac{Y D(1, 1)}{r - \alpha} - I \quad \text{if } Y \geq Y_F$$

Simultaneous Investment

Let us take the opportunity to define the simultaneous investment value as function of the stochastic demand chock Y , denoted by $S(Y)$, as:

$$S(Y) = \frac{Y D(1, 1)}{r - \alpha} - I$$

In the paper, Huisman & Kort use the same notation of Fudenberg & Tirole for simultaneous investment value as $M(Y)$. I prefer the notation $S(Y)$ even because M can confuse with monopoly.

The value of simultaneous investment is important because it is necessary to answer fundamental questions like: (a) what if I deviate from the follower waiting strategy by investing? and (b) I want to become a leader, but what if the rival has the same idea at the same time and both invest simultaneously?

In other words, it is necessary to check if a follower strategy is a Nash-equilibrium and the expected value (or expected losses) if by "mistake" both invest at the same time (this will be important for the calculus of mixed strategies equilibria).

Note that the simultaneous investment can be the *optimal policy* for both if the state of demand is so high that $Y \geq Y_F$, but a mistake in case of $Y < Y_F$. The chart from a numerical example soon will show this issue in a clear way.

Leader Threshold

Without the preemption menace, the firm will invest optimally at the monopoly threshold Y_M . However, due to the menace of preemption, firms cannot wait so long to invest. If one firm wait until $Y = Y_M$, the other firm can invest at $Y_M - \epsilon$, but the first firm could preempt the rival by investing before when $Y = Y_M - 2 \epsilon$, etc. This process stops when one firm has no more incentive to preempt the rival. Firm 1 has incentive to become leader if $L_1 > F_1$ and, most important for firm 1 decision, firm 1 knows that firm 2 also has incentive to become leader if $L_2 > F_2$. So, firm 1 strategy to become leader is to invest when $L_2 = F_2$. However, due the symmetry of the problem, $L_1 = L_2$ and $F_1 = F_2$ for all Y , so that the **leader threshold** is defined as **the Y in the interval $0 < Y < Y_F$** so that the values of leader and follower are equal, i.e.,:

$$Y_L := \{0 < Y < Y_F \mid L(Y) = F(Y)\}$$

There is a proposition in Huisman & Kort showing that this Y (the leader threshold) is **unique**. Due to the symmetry, this threshold is valid for both firms.

The threshold notation Y_L is different of Huisman & Kort. They prefer use Y_P for the threshold that I call Y_L . The subscript "P" comes from "preemption". I prefer the simpler notation of calling Y_L the threshold that the leader will enter in equilibrium.

Numerical Example

Let us illustrate the mathematical development with a numerical example and charts. The base case of the example is the same of the paper of Huisman & Kort.

The **base-case parameters** were (percentage in annual basis):

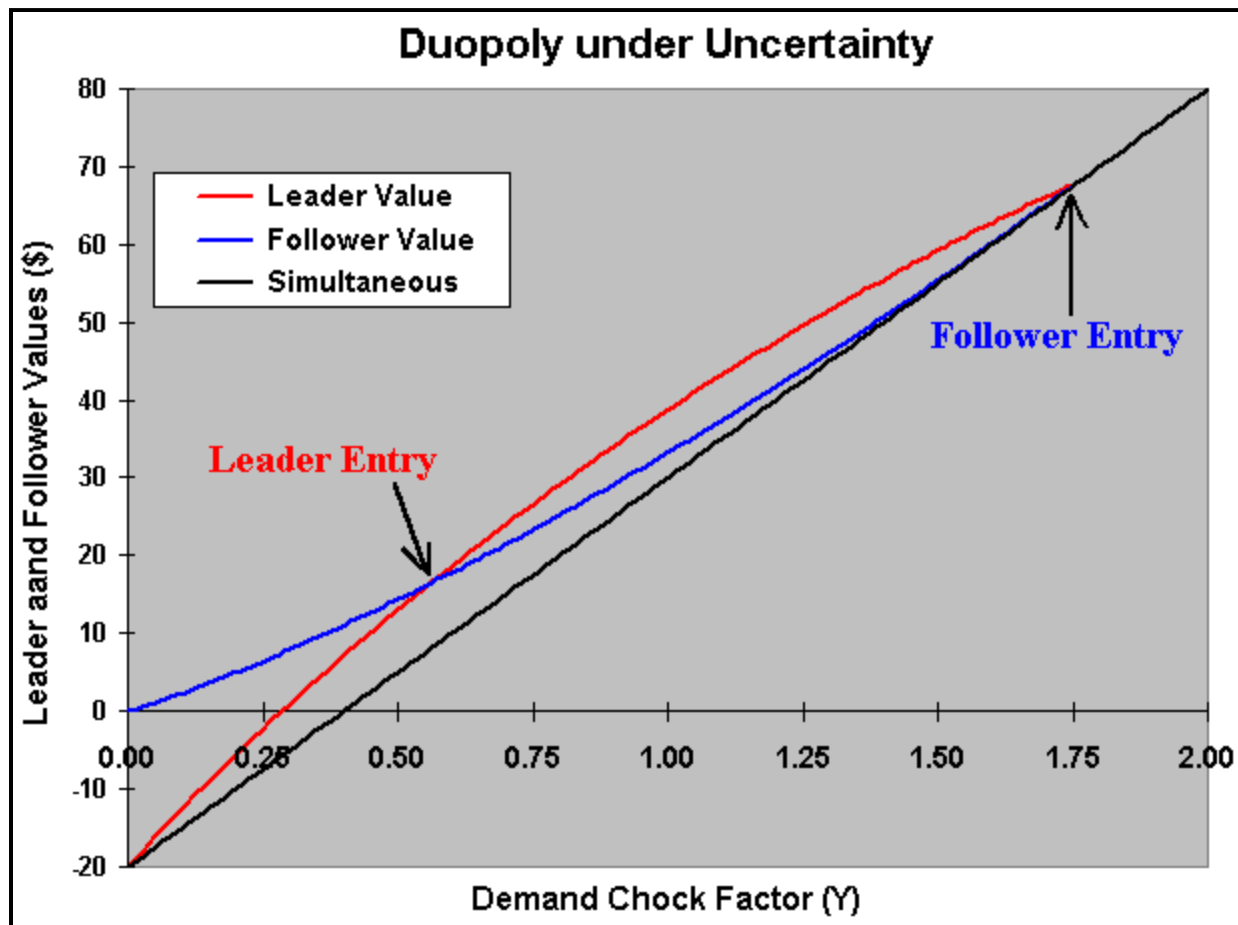
- **Drift $\alpha = 5\%$**
- **Volatility $\sigma = 20\%$**
- **Risk-free discount rate $r = 10\%$**
- **Initial value of demand chock $Y = 1$**
- **Investment $I = 20$ (each firm)**
- **Non-stochastic demand factors:**
 - **$D(0, 1) = 1$**
 - **$D(0, 0) = 2$**
 - **$D(1, 1) = 2.5$**
 - **$D(1, 0) = 4$**

Note that the non-stochastic demand factors obey both the negative externalities and first-move advantage assumptions of the model.

In case of risk-averse firm assumption we can think 5% as the risk-neutral drift in a contingent claims approach.

The chart below presents the leader value, follower value, and the simultaneous exercise value, as functions of the demand stochastic chock Y .

The chart also shows the leader threshold $Y_L = 0.558$ and the follower threshold $Y_F = 1.762$.



4) Mixed Strategies Equilibria

Now, imagine that the initial state of the demand is between Y_L and Y_F . Both firms have incentive to become leader because $L > F$ (see the previous chart to clarify this point). There is no logic to imagine that, without any communication, the other firm will let the rival to become leader so that the probability of simultaneous investment is zero. Both firms will wish the higher leader payoff, but both firms fear the possibility of becoming worse in case of simultaneous investment (a "mistake"). The payoff from simultaneous exercise is lower even than the follower payoff.

Assuming non-communication between the players, the only rational way to treat this problem in the game theory context is by allowing mixed strategies. When communication is allowed - as suggested in Joaquin & Buttler (2000) - there exists the alternative of a *bargain-game* (sharing the surplus $L - F$) with agreement for only one firm to become leader in order to avoid the "mistake" of simultaneous investment.

However, in many situations this kind of agreement is illegal or contrary to accepted convention.

With mixed strategies, firms will rationally calculate the optimal probability to exercise the investment option aiming the leader payoff, but considering the positive probability of simultaneous investment. Firms will play a simultaneous game (with possibly infinite rounds) where the firm i can choose invest

with probability p_i and not invest with probability $1 - p_i$, players $i = 1$ or 2 .

The analysis of mixed strategies in continuous-time *preemption games*, must be performed carefully and using special tools. The passage from discrete-time to continuous-time presents problems when using traditional limit considerations. Fudenberg & Tirole (1985) reported that the usual methods present "loss of information" in this passage, with the continuous case not representing the limit of the discrete case. In addition, with traditional tools many strategies converge with probability 1 to be played at the instant $t = 0$, a non-consistent result.

In order to determine the symmetrical mixed strategies, Huisman & Kort used the same tool applied in Fudenberg & Tirole (1985): they specify "probabilities" named of "*atoms*" $p(\tau)$ that, if positive, indicate cumulative probability of exercise $G_i(\tau)$ equal to 1. So, τ is defined as the first time that some player will exercise the option to invest given that nobody exercise the option before.

This resource is taken from *optimal control literature* (see for example Birge & Louveaux, "Introduction to Stochastic Programming", Springer Verlag, 1997, p.289).

The fundamental idea is that this **control doesn't take time**. Using this analogy, the control here is the result of a simultaneous game that can be repeated. This is like an *instantaneous automatic optimizer*. In this way, a simultaneous game with two players even if repeated infinite times, is played instantaneously (no time consume). This approach determines the probabilities from the mixed strategies and is possible to prove that this is the true limit of the equivalent game in discrete-time.

The simultaneous game, which can be repeated infinite times, at the instant that one (or both) player will exercise the option (that is, at τ), is showed in strategic form below, together with the probabilities of option exercise.

		Firm 2	
		$p_2(\tau)$	$1 - p_2(\tau)$
Firm 1	$p_1(\tau)$	$S(Y(t))$, $S(Y(t))$	$L(Y(t))$, $F(Y(t))$
	$1 - p_1(\tau)$	$F(Y(t))$, $L(Y(t))$	Repeat the game

The firm 1 payoff, V_1 (not yet optimized) is given by:

$$V_1 = p_1 p_2 S + p_1 (1 - p_2) L + (1 - p_1) p_2 F + (1 - p_1) (1 - p_2) V_1$$

The last term means that, in case of repetition we get the payoff V_1 because due to the definition of τ when this game is played (at τ) is certain that some player (or both) will exercise the option to invest. If you don't understand the equation above, an equivalent but perhaps more intuitive way is given by the equation below:

$$V_1 = [p_1 p_2 S + p_1 (1 - p_2) L + (1 - p_1) p_2 F] \cdot [1 + (1 - p_1) (1 - p_2) + (1 - p_1)^2 (1 - p_2)^2 + \dots]$$

The term between the first [.] is the expected payoff of one round in case of definition (investment of one or two players). The second [.] , in red, multiplies that expected payoff and consider the case of definition in the first round (multiplying by one), in the second round [multiplying by $(1 - p_1) (1 - p_2)$], and etc., until the infinite. The red term is just the infinite sum of a geometric progression with ratio lower than one (convergent), so that it is easily calculated. The reader is invited to check out the algebra in order to conclude that the two methods are equivalent.

Hence, the equation for the (non-optimized) payoff from the simultaneous game for the firm i , V_i , $i = 1$ or 2 , is:

$$V_i = \frac{p_i p_j S + p_i (1 - p_j) L + (1 - p_i) p_j F}{1 - [(1 - p_i) (1 - p_j)]}$$

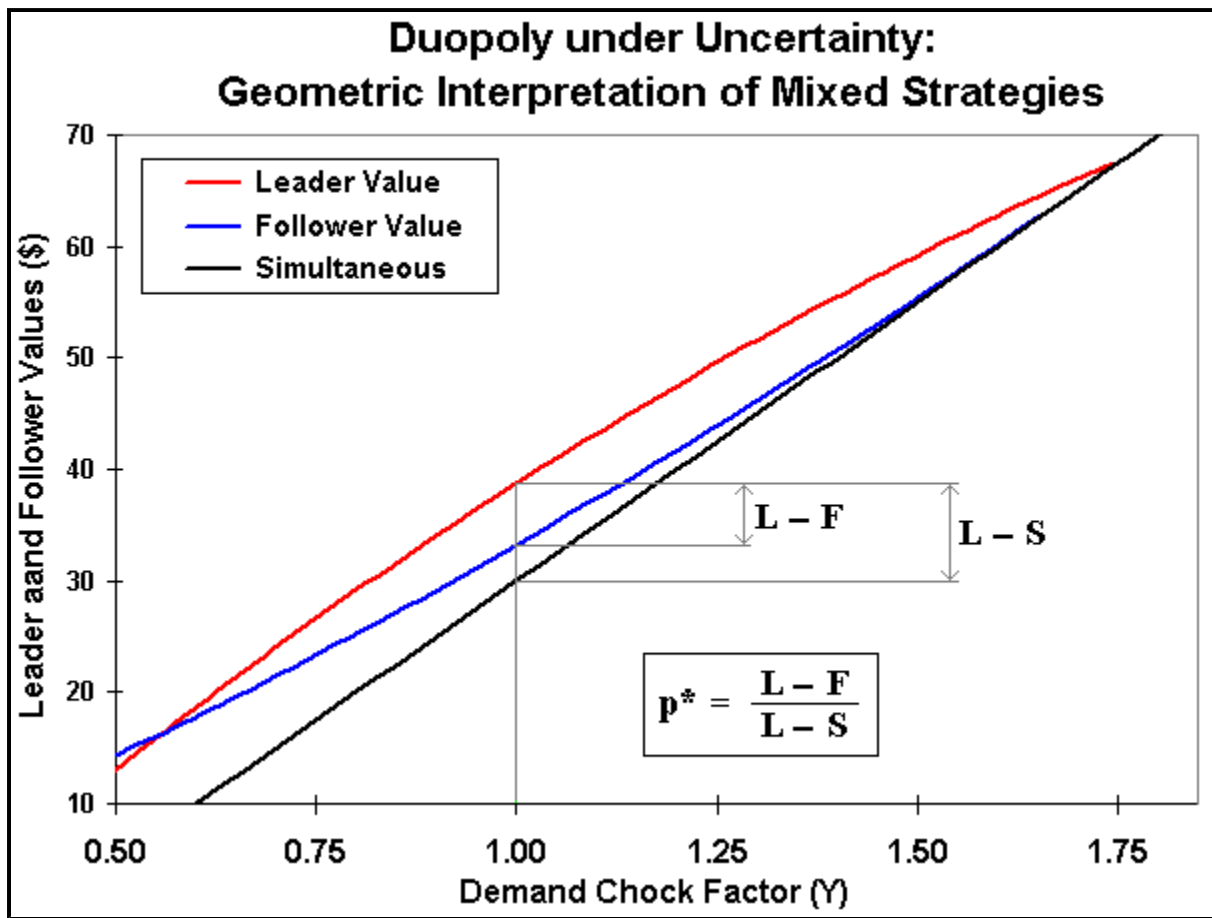
Next, players need to set the optimal probability to exercise the option, that is, the probability that maximizes the expected payoff V_i .

The first order condition for this optimization problem for the firm 1 is $\partial V_1 / \partial p_1 = 0$, given the rival is playing exercise the option with probability p_2 . It is ease to check that the second order condition indicates a maximization problem (second derivative is negative).

Note also that, due to the symmetry of the problem, the optimal probabilities must be equal. That is, $p_1 = p_2 = p^*$. This permits a further simplification. Using the first order condition and the symmetry insight, after some algebra we get the following simple equation for the optimal probability of investing in mixed strategies:

$$p^* = \frac{L - F}{L - S}$$

The chart below presents a zoom in the previous chart (example from the last section) in the region that there exists incentives for both firms to become leader ($L > F$). The figure gives a geometric interpretation for the optimal probability of exercise p^* in mixed strategies.



In the figure above, it is easy to see that when $L = F$ and $L > S$, we have $p^* = 0$. When L tends to both F and S , we have $p^* = 1$. In the mixed strategy equilibrium, the probability p^* can be interpreted as equal the ratio of the possible benefit from preemption ($L - F$) to the range of total possible variation ($L - S$) (note that $L - S = [(L - F) + (F - S)]$).

Now, let us calculate the probability of only one of the firms to exercise the option and the probability of simultaneous exercise in this game. For that, look again the figure with the strategic-form of the simultaneous game and, using a similar reasoning from the payoff, we can write that the probability **pr(one = i)** of only the firm i investing in one round from the simultaneous-game at τ is given by:

$$\mathbf{pr(one = i) = p(\tau) (1 - p(\tau)) + (1 - p(\tau)) (1 - p(\tau)) pr(one = i)}$$

So, the probability **pr(one = i)** is:

$$\mathbf{pr(one = i) = \frac{1 - p(\tau)}{2 - p(\tau)}}$$

Of course, the probability $pr(one = j)$ of only the firm j investing in this simultaneous-game at τ is exactly the same due to the symmetry.

Now, the probability of simultaneous exercise (or **probability of "mistake"**, if $Y < Y_F$), denoted by **pr(two)**, using a similar reasoning is given by:

$$\text{pr(two)} = \frac{p(\tau)}{2 - p(\tau)}$$

Note that the three probabilities sum one, that is, $p(\text{one} = i) + p(\text{one} = j) + \text{pr(two)} = 1$. This is because there is no possibility of no exercise in this simultaneous-game due to the definition of τ .

We can use the previous optimal value that we found for the probability $p(\tau) = (L - F) / (L - S)$, and easily estimate the probabilities of each firm investing and the probability of simultaneous exercise. For example, at Y_L we know that $L = F$ and $L > S$. So, $p(\tau) = 0$. In this case $\text{pr(two)} = 0$ and $p(\text{one} = i) = p(\text{one} = j) = 1/2$.

Note that the probability of simultaneous investment is consistent in the limits when Y tends to Y_F or to Y_L :

- When $Y > Y_F$ is optimal the simultaneous investment. Hence, the (equilibrium) probability of simultaneous investment is 1. In the limit at Y_F for the above equation this probability is one (this limit is not obvious but easily calculated using basic calculus), consistent with the intuition;
- When $Y < Y_L$ the *wait and see policy* is optimal. Hence, the (equilibrium) probability of simultaneous investment is 0. In the limit at Y_L this probability is zero, and again the intuition is consistent with the equation above. Note that there are no jumps in the payoff functions, so it is intuitive that there are no jumps in the exercise probability as well.

Therefore, when the **market births with $Y < Y_L$** there are 50% chances each to become leader and zero probability of "mistake" when Y reach Y_L . This conclusion from Dixit & Pindyck with trivial mixed strategies outcome is correct only in this case, not in general as they claimed. After reaching Y_L we have already defined the leader and the follower and mixed strategy is not necessary anymore. However, only in this case the mixed strategies developed above are not relevant.

For the case with market birthing with $Y_F > Y(t = 0) > Y_L$, *mixed strategy analysis is essential* and there exists a strictly positive probability of mistake as we saw above. Huisman & Kort made this critic and filled the gap by examining the general case with market starting with any feasible value for the demand chock Y .

Denoting the time for Y reaching the thresholds Y_F and Y_L as $T_F = \inf(t \mid Y(t) \geq Y_F)$ and $T_L = \inf(t \mid Y(t) \geq Y_L)$, respectively, we have the following specification for the symmetric mixed strategies equilibria $G_i(t)$, $p_i(t)$:

$$\begin{aligned} G(t) &= 0 \text{ if } t < T_L; \text{ and } G(t) = 1 \text{ if } t \geq T_L \\ p(t) &= 0 \text{ if } t \leq T_L; p(t) = 1 \text{ if } t \geq T_F; \text{ and} \\ p(t) &= \frac{L - F}{L - S} \text{ if } T_L < t < T_F \end{aligned}$$

The symmetric strategies means that the above description of the equilibria is valid for both firms (i.e., $i = 1$ or 2).

5) Collusion Value and Thresholds

Another interesting extension of Huisman & Kort is to consider the possibility of collusion equilibria *without a binding contract* between the firms (without communication). They analyze if there are situations that a *tacit* wait and see policy is equilibrium. Firms could calculate if is the best strategy or not the wait and see until one level $Y = \theta$, when both invest simultaneously - or one firm invest and the other one invest immediately after that.

Collusion only will be *Nash equilibrium* if there is no unilateral incentive to deviate. Deviation means to earn the leader payoff with the other firm choosing optimally to invest much later as follower. We need to calculate the collusion value of one firm waiting to invest simultaneously at the level θ , denoted by $C(Y, \theta)$. In this case each firm will pass from the demand status $D(0, 0)$ directly to $D(1, 1)$. **Collusion will be Perfect-Nash equilibrium only if $C(Y, \theta) \geq L(Y)$ at least for all Y in the interval $(0, Y_F)$.**

With the usual real options steps (Itô's lemma, differential equation, boundary conditions, etc.), it is easy to reach the following value for one firm in collusion:

$$C(Y, \theta) = \begin{cases} \frac{Y D(0, 0)}{(r - \alpha)} + \left(\frac{Y}{\theta}\right)^{\beta_1} \left[\frac{\theta [D(1, 1) - D(0, 0)]}{(r - \alpha)} - I \right], & \text{if } Y < \theta \\ \frac{Y D(1, 1)}{(r - \alpha)} - I, & \text{if } Y \geq \theta \end{cases}$$

The Pareto optimal choice of the argument theta, will maximize the collusion value of each firm at $\theta = Y_C$. In mathematical terms,

$$Y_C = \arg \max_{\theta > Y} C(Y, \theta)$$

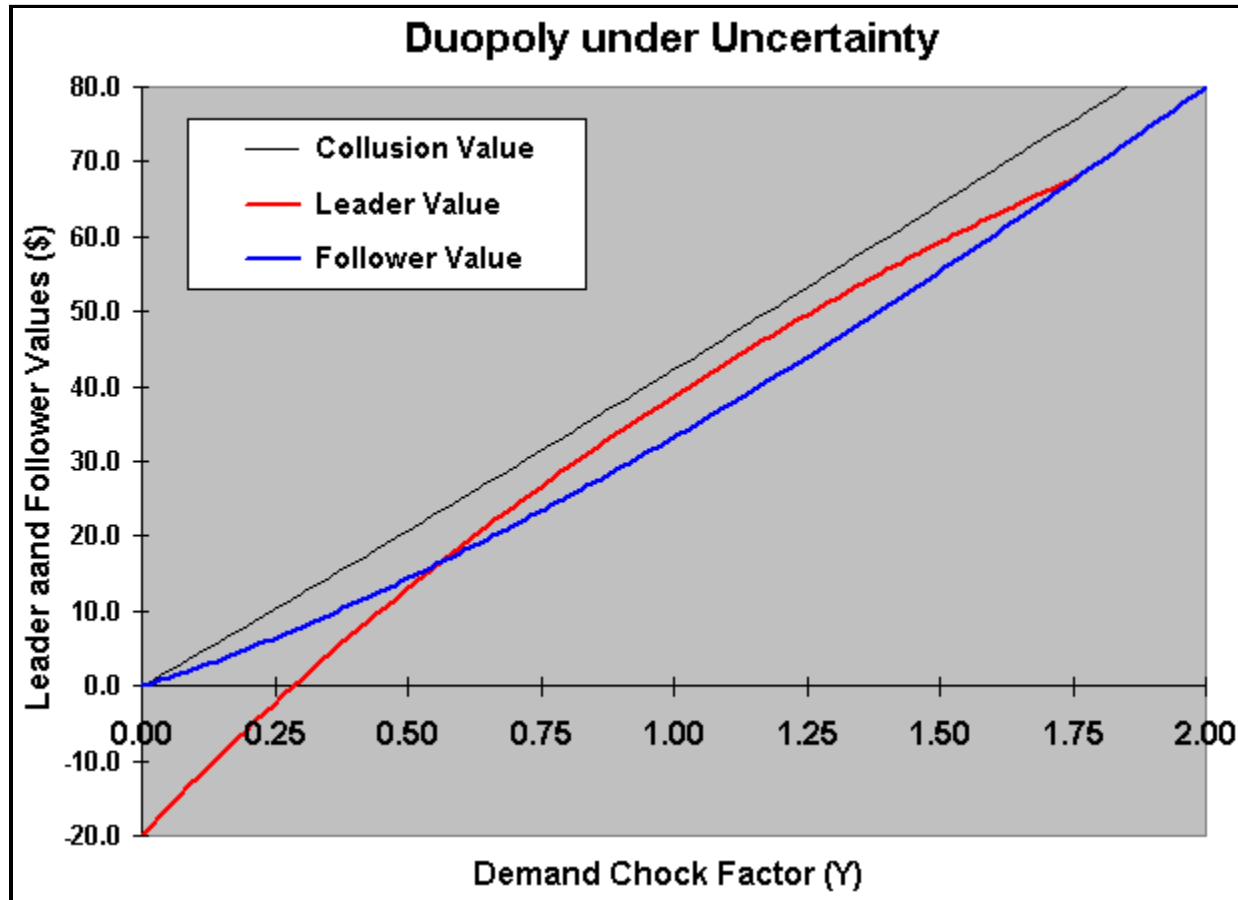
That is, the optimal collusion threshold Y_C (*smooth-pasting* condition at Y_C) is:

$$Y_C = \frac{\beta_1}{\beta_1 - 1} \frac{(r - \alpha) I}{D(1, 1) - D(0, 0)}$$

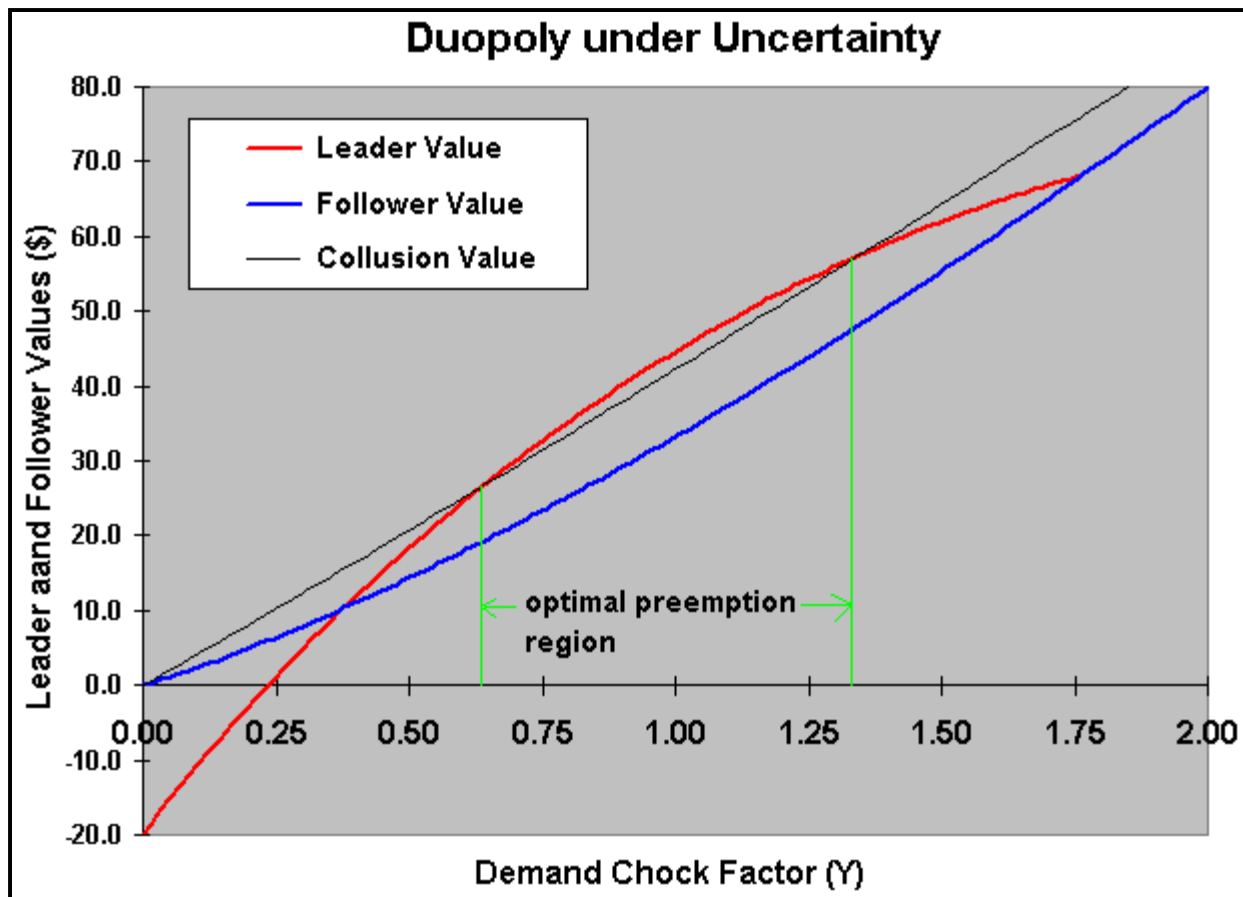
This Pareto optimal is the most probable equilibrium outcome (if collusion is equilibrium) because if one firm invest before, there is a credible menace from the other firm to invest immediately after, obtaining a value lower than $C(Y, Y_C)$.

The chart below presents the leader value, the follower value, and the **collusion value** as functions of the stochastic demand shock Y .

The used parameters were the same of the previous charts. Note that the collusion value is always higher than the leader one, so that the collusion equilibrium is feasible (firms in collusion have **no** incentive to cheating). For the base-case parameters, the collusion value curve smooth-pasting on the simultaneous value line at $Y_C = 5.287$, a very high value so that it is out of the chart.



If the first-mover advantage is sufficient large, the collusion can be destroyed by the preemption. The figure below illustrates this point. Now the value of $D(1, 0) = 5$, which is higher than in previous case, so that now there is a region that preemption is optimal due to the higher value for the leader role when comparing with the collusion value. So, here the firms in collusion have incentive to cheating (collusion is not Nash equilibrium).



6) Equilibria Analysis: Collusion x Preemption

The symmetric preemption equilibria from the last section remain being equilibria (why?). In addition, depending of the parameters we can also have infinite collusion equilibria.

For example, we can choose a $\theta = Y_S < Y_C$ so that there is no incentive to deviate - it is satisfied the condition $C(Y, Y_S) \geq L(Y)$ for all Y in the interval $(0, Y_S)$. By defining $T_S = \inf(t \mid Y(t) \geq Y_S)$ and $T_C = \inf(t \mid Y(t) \geq Y_C)$, we have in this case infinite collusion equilibria by choosing the simultaneous investment at the instant T^* in the interval $[T_S, T_C]$.

As pointed out before, the Pareto equilibrium (at $\theta = Y_C$) is the more probable outcome from this infinite set of collusion equilibria.

As before, the equilibrium strategies are denoted by the pair of cumulative probability of exercise $G_i(t)$ and the instantaneous exercise probability $p_i(t)$, which in this collusion case are:

$$\{ G_i(t) = 0 ; p_i(t) = 0 \} \text{ if } t < T^*$$

$$\{ G_i(t) = 1 ; p_i(t) = 1 \} \text{ if } t \geq T^*$$

An important result for the "new market model", that is, $\mathbf{D(0, 0) = D(0, 1) = 0}$, as in Dixit & Pindyck, the collusion value and the threshold are both equal to the follower value and threshold. This means that **collusion is never equilibrium in the new market model**. The curves of values of collusion and follower coincide, so that *preemption is always the equilibrium scenario in the new market model*. While in the *new market model* the investment timing of the leader does not change the follower profit flow, in a more general model there exists a strategic interaction so that $Y_F < Y_C$ in the more general case - invest earlier to recapture market share lost to the leader, and $Y_C = Y_F$ **for the new market model**.

The Huisman & Kort claims that being the threshold Y_C equal the threshold Y_3 from Dixit & Pindyck (Figure 9.2, p.311), this threshold must be equal to the follower threshold, $Y_3 = Y_C = Y_F$, not higher than Y_F as showed in the book figure. This issue, $Y_3 = Y_C > Y_F$, can occur only in the more general model of Huisman & Kort. I agree with Huisman & Kort.

The general definition if collusion is or not equilibrium depends of the parameters of the problem and it is helped by two Huisman & Kort propositions using functions $f(\beta_1)$ and $g(\beta_1)$.

Define these functions as:

$$f(\beta_1) = \beta_1 \left(\frac{D(1, 0) - D(1, 1)}{D(1, 1) - D(0, 1)} \right) + \left(\frac{D(1, 1) - D(0, 0)}{D(1, 1) - D(0, 1)} \right)^{\beta_1}$$

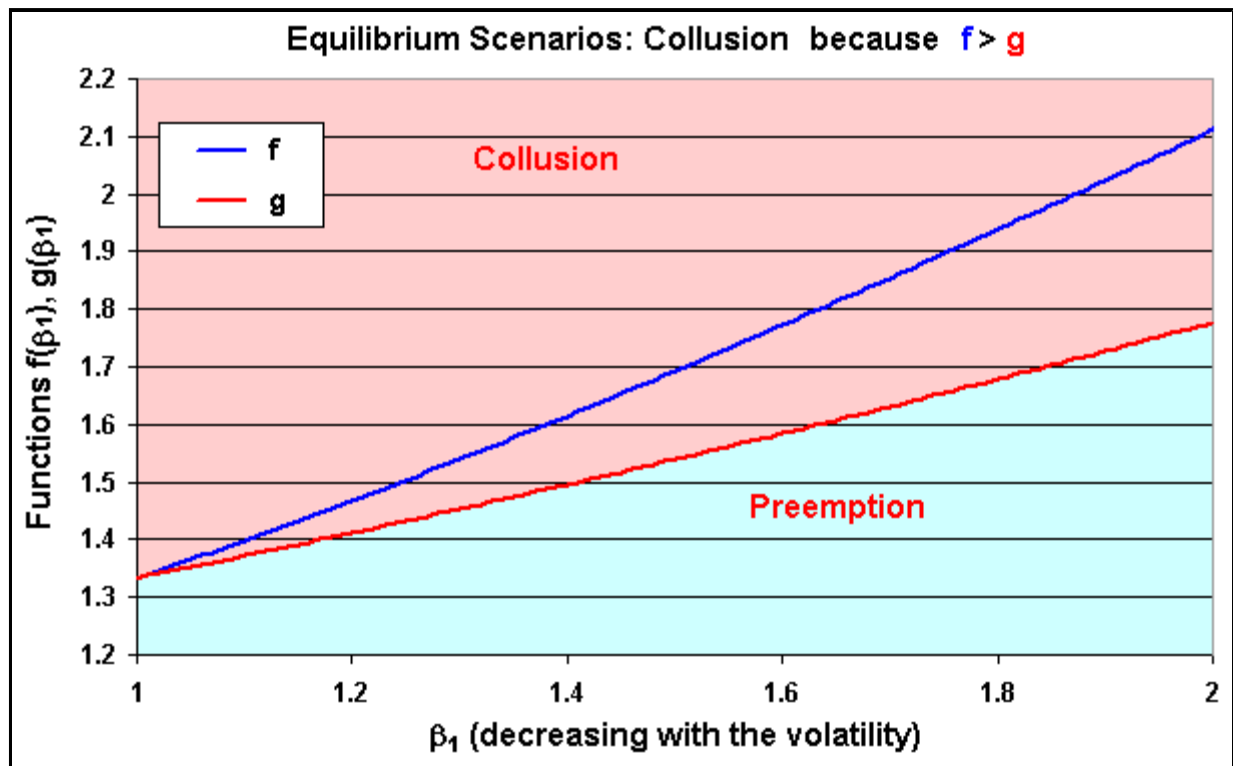
$$g(\beta_1) = \left(\frac{D(1, 0) - D(0, 0)}{D(1, 1) - D(0, 1)} \right)^{\beta_1}$$

The authors proved that there are three possible equilibrium scenarios:

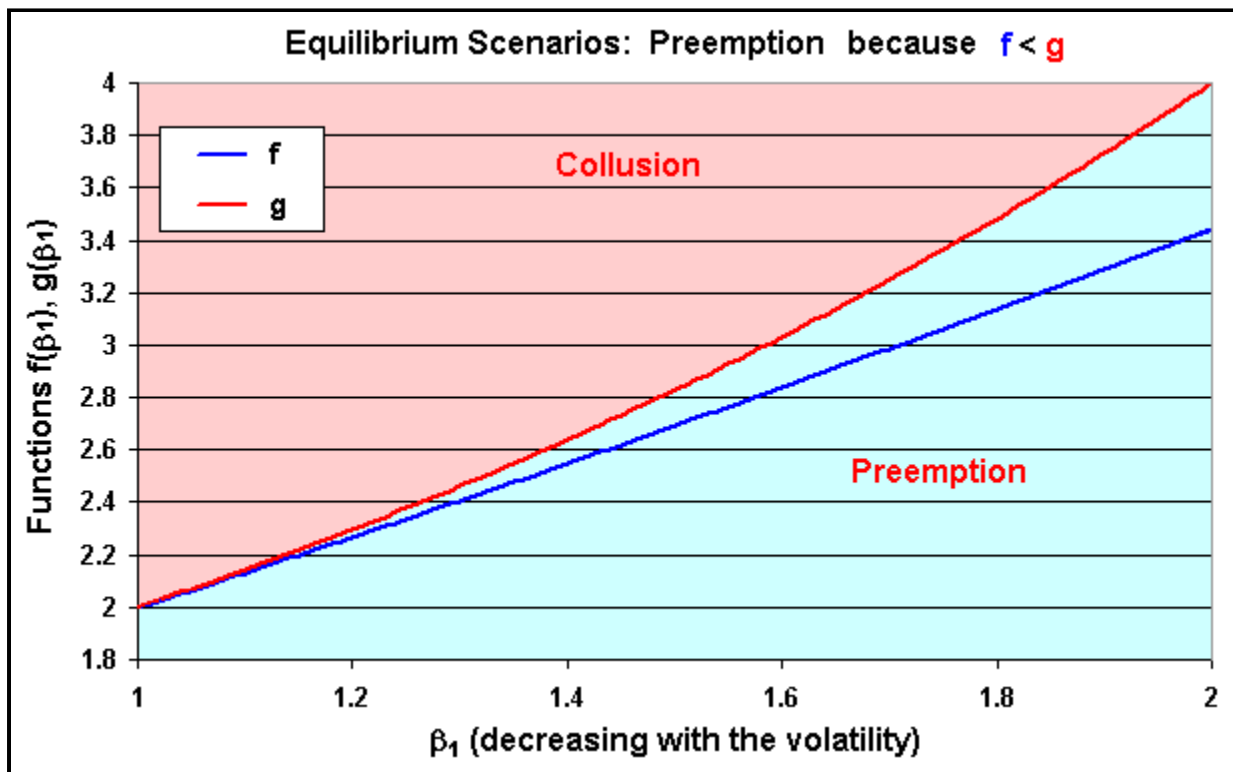
- If $g'(1) \geq f'(1)$, the equilibrium is always of **preemption** type;
- If $f'(1) > g'(1)$ and $f(r/a) \geq g(r/a)$, the equilibrium is always of **collusion** type; and
- If $f'(1) > g'(1)$ and $f(r/a) < g(r/a)$, the equilibrium is of **collusion type for lower values of β_1** and of **preemption type for high values of β_1** .

The next three charts below plot these **functions $f(\beta_1)$ and $g(\beta_1)$** , which define the *collusion and preemption regions*. See the propositions 3 and 4 from the Huisman & Kort's paper for details (or propositions 7.4 and 7.5 from the Huisman's book).

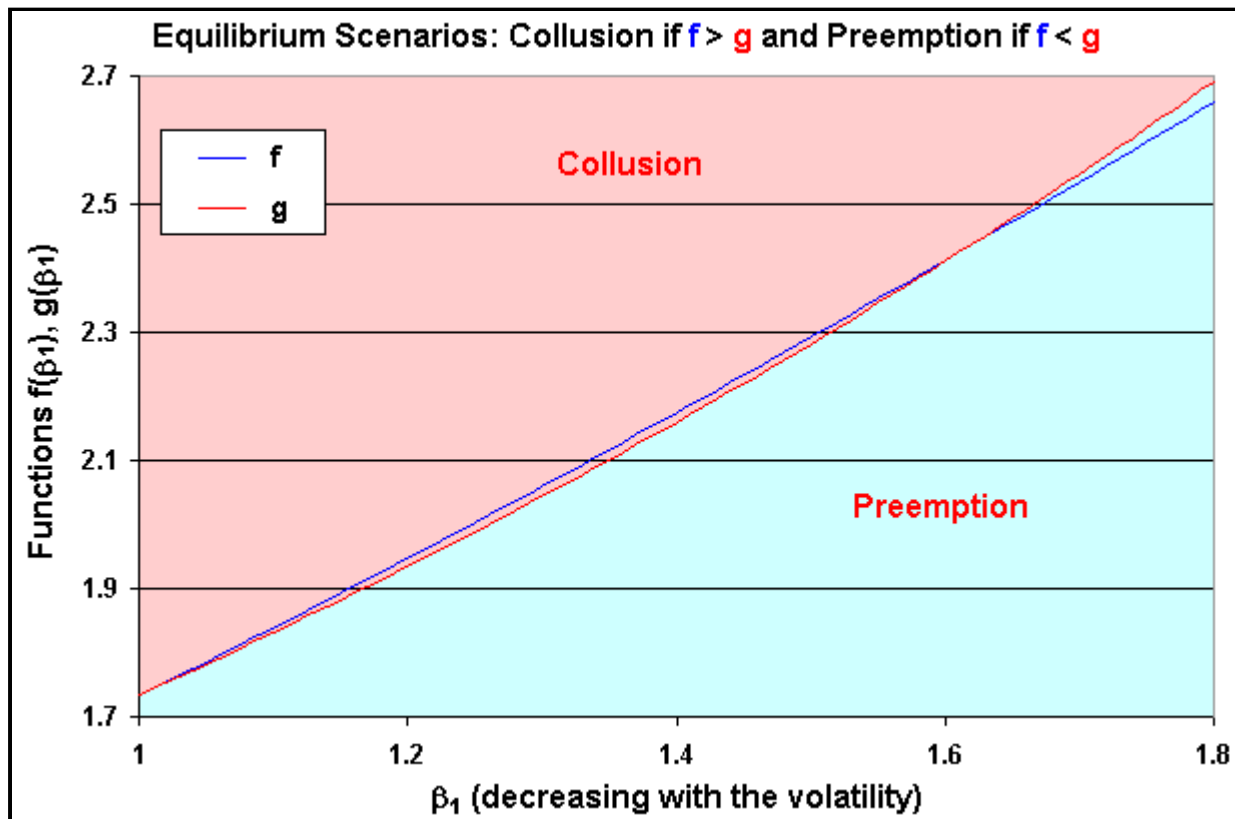
The first chart below is for the base-case used in most of the previous charts with $\mathbf{D(1, 0) = 4}$. In this case, the blue line (function f) is above the function g for all the range of β_1 , indicating that **collusion equilibrium** is the most feasible equilibrium.



The chart below - the second case - is for $D(1, 0) = 5$. In this case, the blue line (function f) is below the function g for all the range of β_1 , indicating *preemption equilibrium*.



The chart below - the third case - is for $D(1, 0) = 4.6$. In this case, the blue line (function f) is above the function g for low values of β_1 - indicating collusion equilibrium, and is below the function g for high values of β_1 - indicating preemption equilibrium.



Remember that β_1 is decreasing with the uncertainty (volatility σ), so that collusion equilibrium is more likely under uncertainty (β_1 near 1) than in the deterministic setting from the seminal paper of Fudenberg & Tirole (1985). The figures above show this issue.

7) Excel Spreadsheet: Contents and Download

The Excel spreadsheet below is the *non-registered version* of a component of the Option-Games Suite. See the page on real options software for the registration fee and the details on how to register. The non-registered version is equal the registered version except that some inputs are fixed and the page with the calculation details is hidden and password protected. However, all the examples and variations discussed above can be reproduced with the non-registered version by changing some non-protected inputs.

The *registered version* gives the freedom to change any input and there is no protection in the registered file, with all sheets unhidden, including the sheet with the detailed calculations. The extensive use of names for cells eases the user to follow the calculations in all the spreadsheet, including the sheet with the calculations details.

The Excel spreadsheet has the following sheets:

- **Input:** enter the deterministic demand parameters $D(., .)$, the parameters from the stochastic process of the stochastic demand process $Y(t)$.
OBS: Some inputs are fixed in the non-registered version.
- **Eq_results:** the equilibrium results, the values and thresholds for the cases of monopoly, leader, follower, collusion, and simultaneous exercise.
OBS: This sheet is non-hidden even in this non-registered version.
- **Propositions:** The Huisman & Kort seven propositions are summarized and checked out for the numerical values inputted to the problem.
OBS: This sheet is non-hidden even in this non-registered version.
- **Chart1, Chart1a, Chart2, and Chart3:** charts which the above figures were derived, presenting the leader, follower, collusion, and simultaneous, values, the functions $f(.)$ and $g(.)$, and a sensitivity analysis of the thresholds with the uncertainty.
OBS: These sheets are non-hidden even in this non-registered version.
- **Detailed_results:** with all the calculus memory used for the results, the charts, the propositions, etc.
OBS: This is the only hidden sheet and password protected, for the non-registered version. In the registered version, there is no password and this sheet is not hidden. The extensive use of names for cells eases the registered user to follow the calculations in this sheet.

The non-registered version can be freely downloaded below and used for educational purposes:

◆ **Download** [Excel file named duopoly_hui-kort_non-reg.xls](#) with 791 KB

This spreadsheet is part of the software pack *Option-Games Suite*. The **registered** version the sheet of **calculus is non-protected and the registered reader can see the calculation details**.
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