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Use of Simulation and Real Options Applications in Natural Resources/Energy

Real Options Valuation in the Modern Economy
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Presentation Outline

- Introduction and overview of Monte Carlo (MC) simulation applied to finance.
  - MC concept. Efficient sampling (MC x quasi-MC).
  - Real x Risk-Neutral simulation of stochastic processes.
  - Equations for simulation of continuous-time stochastic processes: exact and approximate discretizations.
  - MC simulation for European and for American options.
- Applications of MC simulation in real options.
  - Flex-Fuel plant (input flexibility: fuel oil x coal).
  - The biodiesel project.
  - The Bolivia-Brazil gas pipeline tariff case.
  - Learning options + timing options: combining technical and market uncertainties in oilfield development.

Monte Carlo Simulation: Introduction

- Monte Carlo simulation is an increasing popular method to value complex derivatives, including real options.
  - The MC method solves a problem by simulating directly the physical process, so that it is not necessary to write down the differential equations that describe the system behavior.
  - It is a very flexible tool to handle many specific details of real life problems, including many boundary and other system restrictions and many source of uncertainties.
  - In short, it is the antidote to both the *curse of dimensionality* and the *curse of modeling* that plague complex real life problems.
- The MC idea and name are attributed to S. Ulam and N. Metropolis, respectively, in the Manhattan Project at Los Alamos during the World War II times.
  - The first Monte Carlo paper, named "The Monte Carlo Method“, by Metropolis & Ulam, was published in 1949 in the *Journal of the American Statistical Association*.

Monte Carlo Simulation at Work

- MC method consists basically of:

(a) specify the input variable distributions (including sequence of distributions along the time, i.e., stochastic processes) and eventual correlations/dependences;

(b) sample these input distributions;

(c) do mathematical operations with the sampled inputs (+, –, *, ^, /, exp[.], etc.) to calculate the output from these samples;

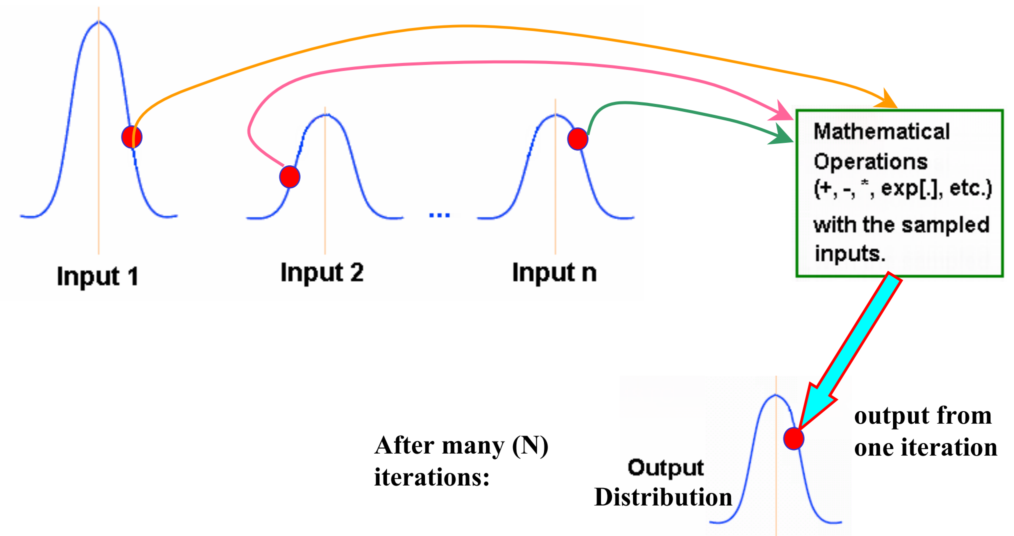
(d) repeat the previous steps N times, generating N outputs; and

(e) calculate the mean and other probabilistic properties from the resulting output distribution.

- The figure in the next slide illustrates this procedure.

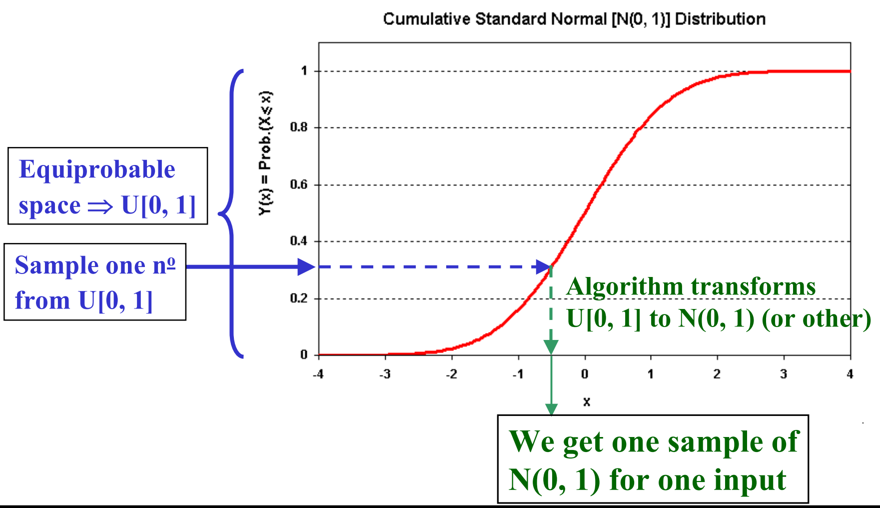
Monte Carlo Simulation at Work

- The MC method is illustrated with the figure below:



MC Simulation: Sampling Process

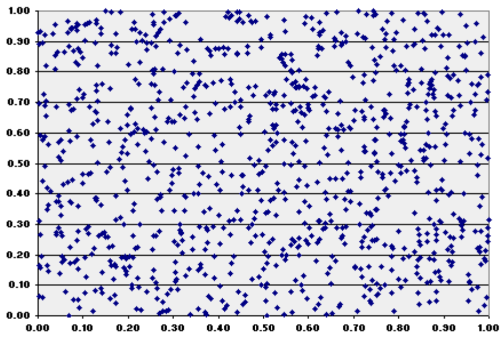
- The simulation numerical quality depends on the sampling process. Figure below (standard Normal) shows that in general we need only the uniform distribution for the interval [0, 1].
  - Algorithms transform the U[0, 1] sample to other distribution sample.



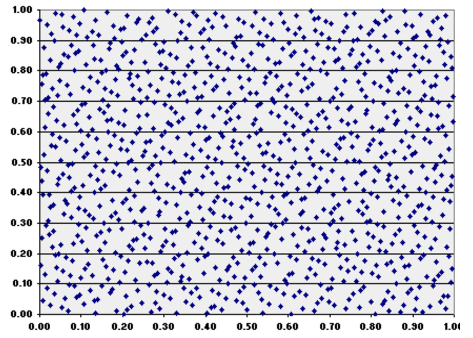
Generating U[0, 1]: Pseudo x Quasi-Random

- **The simulation numerical accuracy depends on the quality of U[0, 1] generation. In order to generate U[0, 1] the traditional approach is the *pseudo-random* one, e.g., Excel function Rand().**
  - **A better approach is to generate quasi-random numbers (sequences of *low discrepancy*). The figure below compares the two approaches for the bi-dimensional uniform. Note that quasi-random case shows a more evenly dispersed points (less clustered than pseudo-random).**

Pseudo-Random



Quasi-Random



Real x Risk-Neutral Stochastic Processes

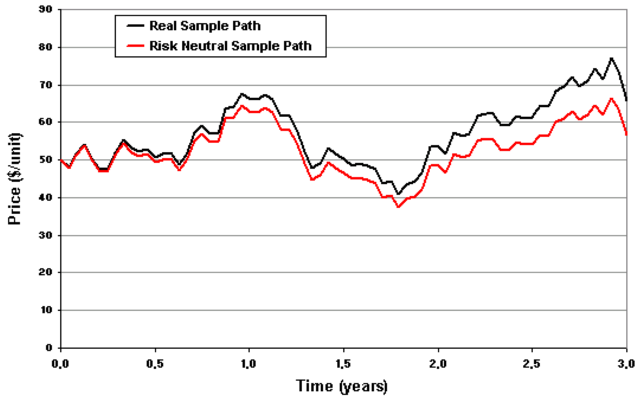
- **We can simulate either the real stochastic process or the risk-neutral stochastic process. The difference is a risk-premium π subtraction from the real drift α.**
  - **Risk-neutral simulation is used for derivatives pricing because we don´t know (or is hard) the derivative´s risk-adjusted discount rate.**
    - **So we penalize the drift and so the distribution (lower mean), a *martingale change of measure*, in order to use the risk-free discount rate for the derivative.**
  - **Real drift = α $\Rightarrow$ Risk-neutral drift = $\alpha - \pi$ = $r - \delta$**
- **For the geometric Brownian motion (GBM), used in Black-Scholes-Merton, the real and risk-neutral GBMs are:**

$$\frac{dP}{P} = \alpha \, dt + \sigma \, dz$$ ⇒ **Real GBM.**

$$\frac{dP}{P} = (r - \delta) \, dt + \sigma \, dz'$$ ⇒ **Risk-Neutral GBM.**

Real x Risk-Neutral Stochastic Processes

- A typical sample-paths for both real and risk-neutral GBMs (with the same stochastic shocks) is showed: the difference is π.



- While risk-neutral simulation is used to price derivatives, real simulation is useful for *planning purposes* (e.g., if wait and see is optimal, what is the probability of option exercise?) and for *risk-analysis* (e.g., value-at-risk estimation) & hedging.

Equations for Stochastic Process Simulation

- Some stochastic processes (not all) admit *exact discretization*, i.e., numerical precision independs of the time-step length.
  - This is particularly useful for real options, because we work with long time to expiration, e.g., we can use Δt = 1 year without losing precision.
- The exact discretization equations to simulate both the real and risk-neutral geometric Brownian motions are, respectively:

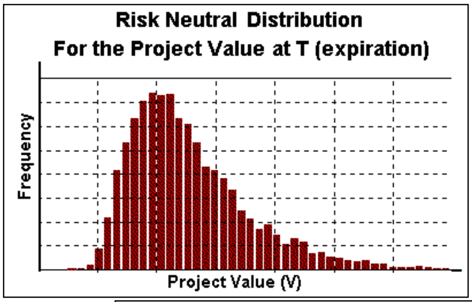
$$P_t = P_0 \exp\{(\alpha - 0.5\,\sigma^2)\,\Delta t + \sigma\, N(0, 1)\sqrt{\Delta t}\,\}$$

$$P_t = P_0 \exp\{(r - \delta - 0.5\,\sigma^2)\,\Delta t + \sigma\, N(0, 1)\sqrt{\Delta t}\,\}$$

  - The difference is the drift. Sampling N(0, 1) n times, we get n outputs P_t.
- Stochastic processes with exact discretizations include mean-reversion. See: www.puc-rio.br/marco.ind/sim_stoc_proc.html
- We can simulate the entire GBM path or only at the expiration (European options). The European options can be calculate by simulation and compared with the Black-Scholes analytic result.

European Call Valuation by Simulation

- If the underlying asset V is the operating project and I is the exercise price (investment), the visual equation for European real option is:



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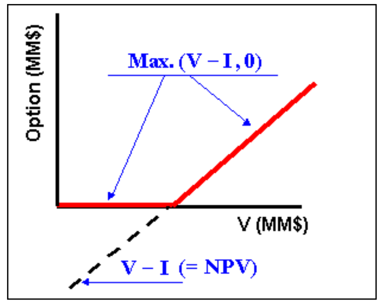

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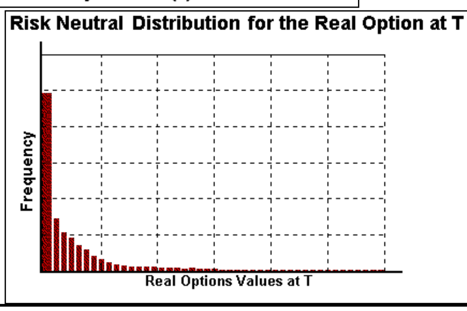
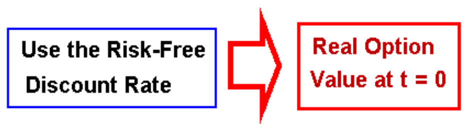




European & American Real Options

- Valuation by simulation of European style real options is very easy because the decision rule is known, e.g., Max[V - I, 0] @ T.
- American real options are much harder to solve by simulation because the optimal decision rule is much more complex:
  - The optimal decision rule for American option is the threshold curve, which is calculated *backwards* whereas simulation is *forward* looking.
- However, if we know in advance the threshold curve or if we combine the MC simulation with one optimization method, we can evaluate American real options by simulation.
  - Since the nineties, there is a growing literature on new methods to evaluate American options by simulation. The best known paper is Longstaff & Schwartz (2001), but there are more than 20 methods.
- Knowing the threshold curve *after* a learning option exercise, we can evaluate complex learning + timing options with MC.
  - We´ll see later an oilfield development case with learning (Dias, 2002) combining technical and market uncertainties with MC simulation.

European Real Options by Simulation

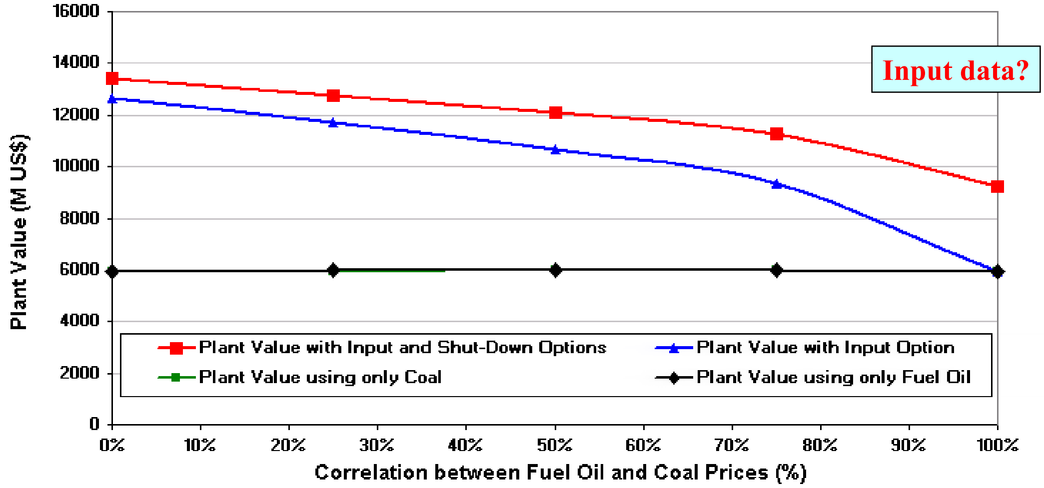
- **There are many practical problems that we can apply the European option valuation by Monte Carlo simulation, mainly *sequence of European real options* (e.g., calls on a *basket* of assets).**
  - **This is best way to valuate projects with flexible inputs and/or flexible outputs, because at specific decision dates (ex.: every month) the firm has to decide the best mix of inputs and outputs for the next operational period (to maximize the payoff, e.g., for the next month).**
  - **We´ll see some real life cases. The idea is to simulate the risk-neutral stochastic processes for the inputs and outputs prices, which are not necessarily GBMs (e.g., could be mean-reversion with jumps).**
  - **In addition, the exercise payoff function can be very complex, with many real life details (e.g., one input is not available in the first year or in certain months; a minimum quantity of one input must be used due to a contract commitment, etc.).**
  - **MC simulation plugged into a spreadsheet is very flexible to handle multiple/complex stochastic processes and complex payoff functions.**

Flex-Fuel Plant with & without Shut-Down Option

- **One firm is going to invest in a energy consuming plant. There are three energy technology alternatives:**
  - **Plant using only oil fuel; plant using only coal; and flex-fuel plant, i.e., plant with (costless) input flexibility (oil or coal).**
  - **We'll see also the flex-fuel plant with costless *shut-down option*.**
- **What are the plant values in each case considering that oil fuel and coal follow correlated mean-reversion processes?**
  - **The answer gives an idea of the maximum value that a firm is willing to pay for the (more expensive) flex-fuel technology.**
  - **Positive correlation decreases the option value, but it is necessary a (unlikely) very high correlation for the input option be negligible.**
- **What is the effect of the costless shut-down option?**
  - **This option can be very important. There are contract implications.**
- **MC simulation answers easily these questions.**
  - **This is a *sequence of European options* (choose the maximum payoff at each operational decision date). The next slide shows an example.**

Flex-Fuel Plant, Correlation & Flexibility Value

- **The chart shows a numerical example with mean-reversion for both oil fuel and coal, for different correlations.**
  - **The chart numerical values were obtained with MC simulation.**
  - **Plant values using only one input (without options) are ~ the same.**

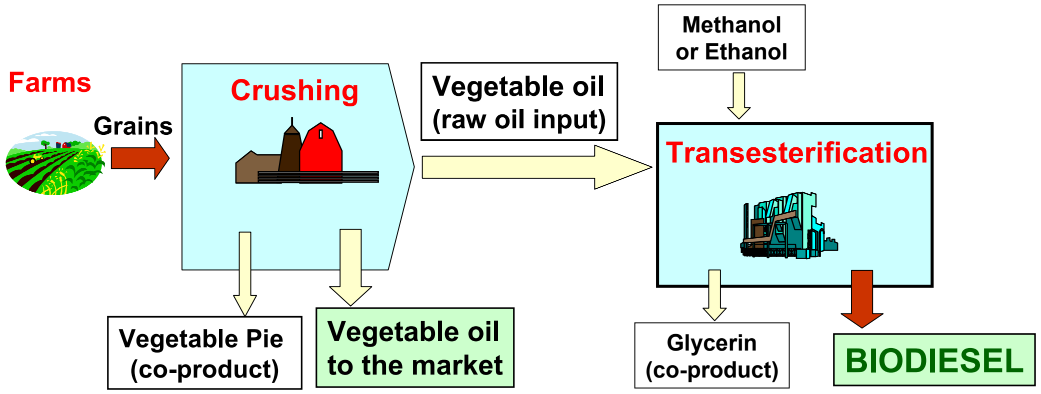


Real Life Application: Biodiesel Project

- **Biodiesel fuel for diesel engines has low emission advantage and is produced from vegetable oil or animal fat by the chemical process of transesterification with alcohols.**
  - **Commercial biodiesel production in US started in late 1990's.**
  - **Biodiesel as fuel additive, will be obligatory in Brazil in 2008.**
- **We are considering only multi-vegetable biodiesel plants.**
  - **So, there is *input flexibility* to choose the vegetable that maximize the project value. Real options is the natural tool to evaluate this.**
  - **Some Braziliam vegetable considered were soybean, cotton, castorbean, pinion (jatropha curcas), uricury syagrus palm, etc.**
  - **In addition, there is input reagent flexibility: methanol or ethanol.**
  - **The vegetables price (and their oils) and the alcohols are commodities and oscillate in the market.**
    - **We use stochastic processes to model these uncertain prices.**

Biodiesel Plant, Inputs and Outputs

- **A biodiesel plant has two main units:**
  - **The *crushing* unit, the vegetable grain is crushed generating raw oil and residue (pie). Raw (vegetable) oil is the main revenue.**
  - **The *transesterification* unit, that uses raw vegetable oil (cost) and reagent (alcohol), generating biodiesel plus residuals .**
  - **The figure below shows the biodiesel plant and its inputs/outputs.**



Biodiesel Project: The Value of Input Flexibility

- **Petrobras biodiesel business format: owner of *both units*, (crushing and transesterification). Why crushing unit?**
  - **In order to guarantee the raw oil quality; and**
  - **In order to *capture the flexibility* (real option) *value* in choosing the vegetable grain input.**
- **This flexibility is modeled as a *sequence of European call options on maximum of several risky assets*:**
  - **At each period the biodiesel plant choose the vegetable(s) and reagent combination that maximizes the profit in that period.**
  - **We performed Monte Carlo simulations for the stochastic processes of the input prices (several grains, vegetable raw oils, methanol, ethanol) and the output prices (biodiesel = diesel, residues, and vegetable oils to the market).**
    - **Difficulties to estimate some stochastic process parameters (lack of data).**
  - **The flexibility (real options value) added a significant and decisive value for biodiesel project economic feasibility.**

Jump to conclusions?

Bolivia-Brazil Gas Pipeline Tariff Case

- In 2000/2001, arbitrated by ANP, took place a dispute between TBG (controlled by Petrobras) and entrants (BG and Enersil) on the Bolivia-Brazil gas pipeline tariff.
  - They wanted pay a tariff value equal to the *ship-or-pay* tariff, but without the ship-or-pay obligation!
    - The entrants had more flexibility: if gas demand drops they can leave the pipeline without paying ship-or-pay tariff. If demand rises, they profit by signing shipments contracts.
    - We argued that the entrants flexibility has value so that their tariff shall be higher (option premium) than ship-or-pay one.
- What is the fair flexibility premium for this tariff?
- The answer is: given the ship-or-pay tariff, a fair "flexible" tariff makes a firm indiferent between paying the cheaper ship-or-pay or paying the more expensive tariff but with flexibility to leave (ship-or-pay plus a positive premium).

Bolivia-Brazil Gas Pipeline Tariff Case

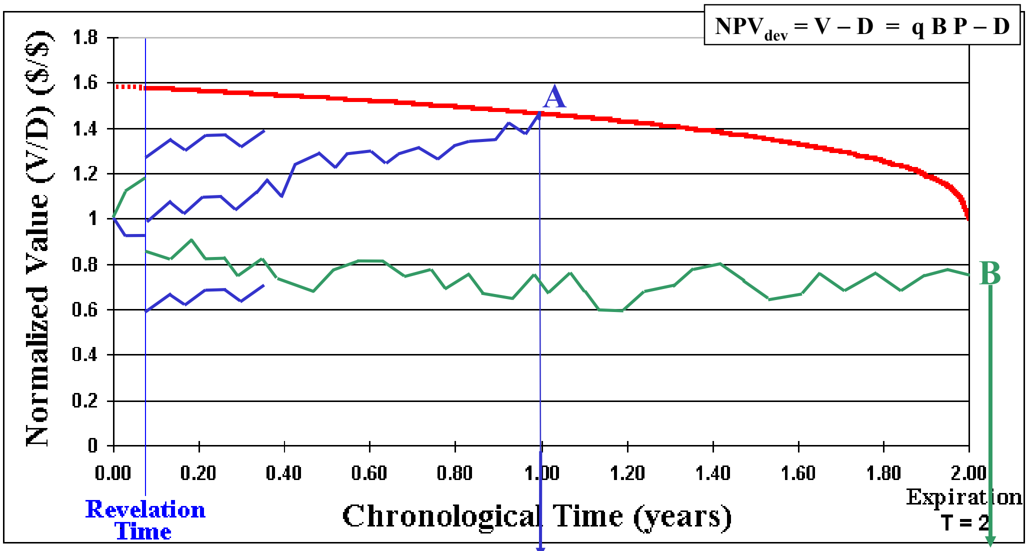
- To estimate the fair flexibility premium for the non-ship-or-pay tariff, we performed a MC simulation of the Brazilian gas market for the term in dispute (3 y.)
  - By using time-series, we saw that *geometric Brownian motion* was a good model for the gas demand in the next three years.
  - We estimated the GBM parameters (mainly the volatility).
  - We set standard contracts of 100,000 m3/d, with one year term. Entrants exercise the options to sign contracts in case of *excess of demand* (demand higher than the current contracted volume). If demand drops, these contracts are not renewed.
  - We simulated the competition about the expected excess of demand *share* that could be captured by entrants and others.
  - We calculated the present values for a firm using ship-or-pay and flexible tariff. <u>The fair tariff equals these present values</u>.
  - Result: We estimated a 20% premium. ANP determined 11%.

Learning + Timing Options in Oilfield Development

- **This case was presented in Dias (2002):**
  - **One oilfield has remaining technical uncertainty about the reserve volume (B) and the reserve productivity (quality q).**
  - **In addition, long-term oil prices (P) and the development investment (D) are uncertain and follow correlated GBMs.**
  - **The development exercise payoff is NPV = V – D = q B P – D**
- **We can invest in information before developing the oilfield.**
  - **There are k alternatives of investment in information (learning options) with different costs, time-to-learn and revelation power (capacity to reduce technical uncertainty).**
  - **Investment in information reveals new expectations about oil reserve volume (B) and quality (q). Technical uncertainty is modeled w/ *revelation* (conditional expectation) *distributions*.**
- **The next slide illustrates the valuation approach with MC.**

Real Options Valuation with Investment in Information

- **M.C. simulation combining market (oil price) and technical uncertainties**



F(t = 0) =
= F(t=1) * exp (– r*t)

Present Value (t = 0)

Option F(t = 1) = V – D

F(t = 2) = 0

Expired Worthless

Conclusions

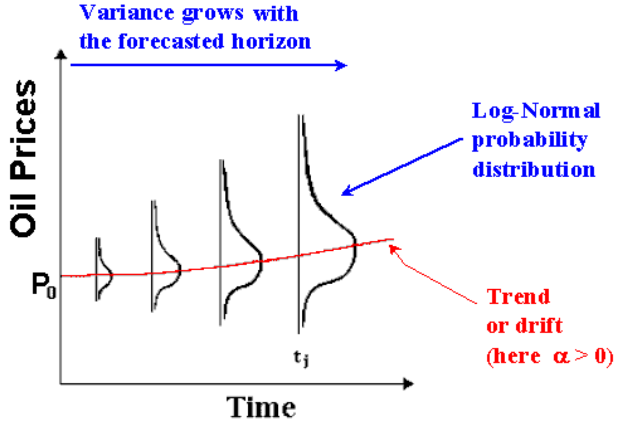
- Monte Carlo simulation is a very flexible tool for real options valuation of complex real life projects.
  - We can easily combine many sources of uncertainties and include many real life restriction details and complex payoffs.
  - We discussed some MC issues like sampling (pseudo x quasi-random simulation), real x risk-neutral stochastic processes, and European x American real options problems.
  - We saw a typical example of a plant value with *input flexibility*.
- We saw two European real option cases using MC:
  - The Brazilian (real life) projects: the Biodiesel project and the Bolivia-Brazil gas pipeline tariff dispute.
- We saw also a more complex (American) MC case:
  - The learning + timing options in oilfield development.
- Thank you very much for your time!

APPENDIX

SUPPORT SLIDES

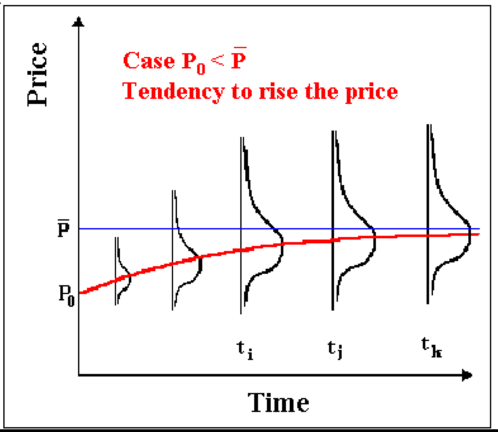
Stochastic Processes for Oil Prices: GBM

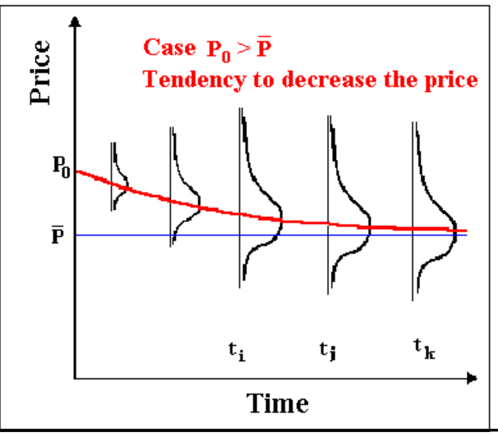
- **Like Black-Scholes-Merton equation, the classical model of Paddock et al uses the popular Geometric Brownian Motion**
  - **Prices have a log-normal distribution in every future time;**
  - **Expected curve is a exponential growth (or decline);**
  - **The variance grows with the time horizon (unbounded)**



Mean-Reverting Process

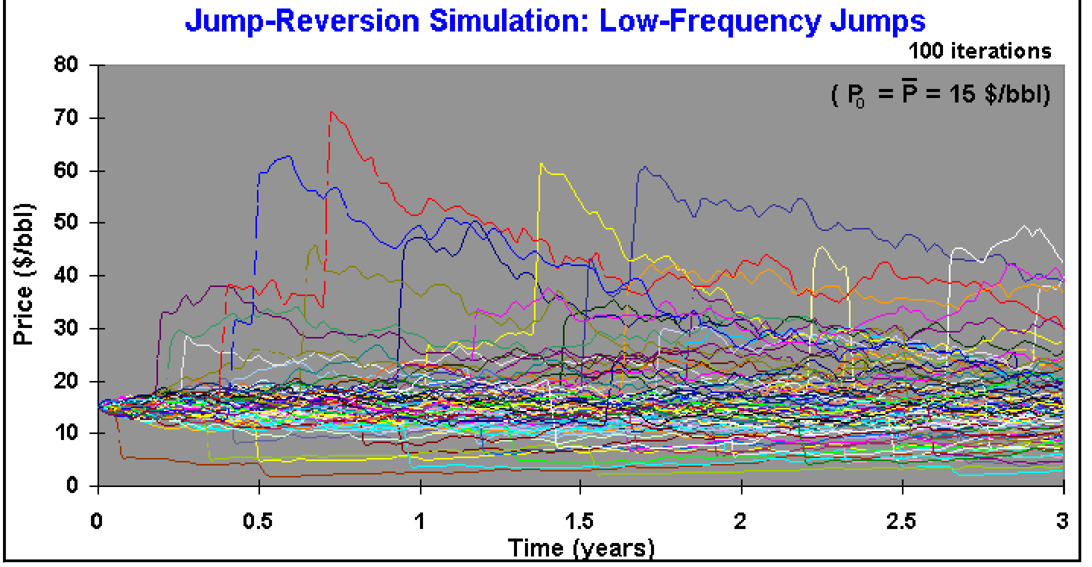
- **In this process, the price tends to revert towards a long-run average price (or an equilibrium level) $\bar{P}$.**
  - **Model analogy: *spring* (reversion force is proportional to the distance between current position and the equilibrium level).**
  - **In this case, variance initially grows and stabilize afterwards**





Mean-Reversion + Jumps: MC *Sample Paths*

- **Chart shows 100 sample paths from MC simulation of mean reversion plus jumps. The starting price and the long-run equilibrium price were $15.**



VBA Code for Black-Scholes-Merton by Simulation

- **Download the Excel file with unprotected VBA code, Visual Basic for Application, at: www.puc-rio.br/marco.ind/xls/qmc_black_scholes.xls**

```
Option Explicit
Option Base 1

Sub QMC_BlackScholesMerton()

    Dim V As Double
    Dim K As Double
    Dim r As Double
    Dim d As Double
    Dim sig As Double
    Dim T As Double
    Dim nsim As Long
    Dim trendRN As Double
    Dim sigraiz As Double
    Dim Xj() As Double
    Dim i As Long
    Dim N01() As Double
    Dim Vt() As Double
    Dim Fsum As Double
    Dim QMC_BSM As Double

    Randomize
    Range("start_time").Value = Now()

    V = Range("P0_")
    K = Range("D0_")
    r = Range("r_")
    d = Range("delta")
    sig = Range("sigma")
    T = Range("T_")
    nsim = Range("NSimul")

     trendRN = (r - d - ((sig * sig) / 2)) * T
     sigraiz = sig * Sqr(T)
     Fsum = 0

     ReDim Preserve Xj(1 To nsim)
     ReDim Preserve N01(1 To nsim)
     ReDim Preserve Vt(1 To nsim)
            For i = 1 To nsim
                Xj(i) = CorputBase2(i)
                N01(i) = Moro_NormSInv(Xj(i))
                Vt(i) = V * Exp(trendRN + (sigraiz * N01(i)))
                    If Vt(i) >= K Then
                    Fsum = Fsum + (Vt(i) - K)
                    Else
                    End If
            Next i
    QMC_BSM = (Fsum / nsim) * Exp(-r * T)
    Application.StatusBar = False
    Range("RealOpt").Select
    ActiveCell.FormulaR1C1 = QMC_BSM
    Range("Finish_time").Value = Now()
End Sub
```

- **This function call other functions: the generation of quasi-random numbers and Normal inversion functions (next slide).**

VBA Code for Black-Scholes-Merton by Simulation

- **The codes below are necessary complements: generation of quasi random numbers (CorputBase2) and Normal inversion (Moro).**

```
Function CorputBase2(N As Long) As Double
'   Returns the equivalent first
' van der Corput sequence number
    Dim c As Double, ib As Double
    Dim i As Long, n1 As Long, n2 As Long
    n1 = N
    c = 0
    ib = 1 / 2
    Do While n1 > 0
        n2 = Int(n1 / 2)
        i = n1 - n2 * 2
        c = c + ib * i
        ib = ib / 2
        n1 = n2
    Loop
    CorputBase2 = c
End Function
```

```
Function Moro_NormSInv(u As Double) As Double
'    Calculates the Normal Standard numbers given u,
'   the associated uniform number (0, 1)
'    VBA version of the Moro's (1995) code in C
'    Option Base 1 is necessary to be declared before
'   this function for vector elements positioning to work
    Dim c1, c2, c3, c4, c5, c6, c7, c8, c9
    Dim X As Double
    Dim r As Double
    Dim a As Variant
    Dim b As Variant
    a = Array(2.50662823884, -18.61500062529, 41.39119773534, -25.44106049637)
    b = Array(-8.4735109309, 23.08336743743, -21.06224101826, 3.13082909833)
    c1 = 0.337475482272615
    c2 = 0.976169019091719
    c3 = 0.160797971491821
    c4 = 2.76438810333863E-02
    c5 = 3.8405729373609E-03
    c6 = 3.951896511919E-04
    c7 = 3.21767881768E-05
    c8 = 2.888167364E-07
    c9 = 3.960315187E-07
    X = u - 0.5
    If Abs(X) < 0.42 Then
        r = X ^ 2
        r = X * (((a(4) * r + a(3)) * r + a(2)) * r + a(1)) / _
        ((((b(4) * r + b(3)) * r + b(2)) * r + b(1)) * r + 1)
    Else
        If X > 0 Then r = Log(-Log(1 - u))
        If X <= 0 Then r = Log(-Log(u))
        r = c1 + r * (c2 + r * (c3 + r * (c4 + r * (c5 + r * _
        (c6 + r * (c7 + r * (c8 + r * c9)))))))
        If X <= 0 Then r = -r
    End If
    Moro_NormSInv = r
End Function
```

More on Simulation of Stochastic Processes & Quasi-Monte Carlo

- **See equations, discussion and Excel spreadsheets on stochastic processes simulation at:**

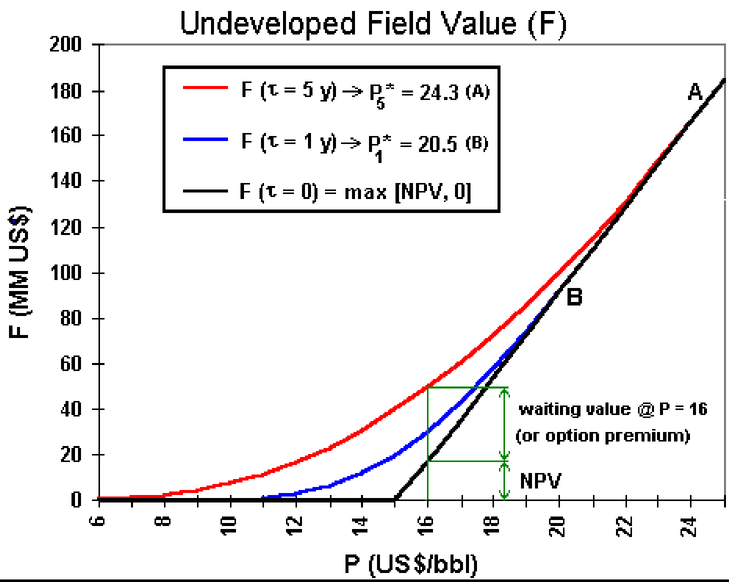
  www.puc-rio.br/marco.ind/sim_stoc_proc.html

- See equations, **discussion and Excel spreadsheets on quasi-random numbers (quasi-MC simulation) at:**

  www.puc-rio.br/marco.ind/quasi_mc.html

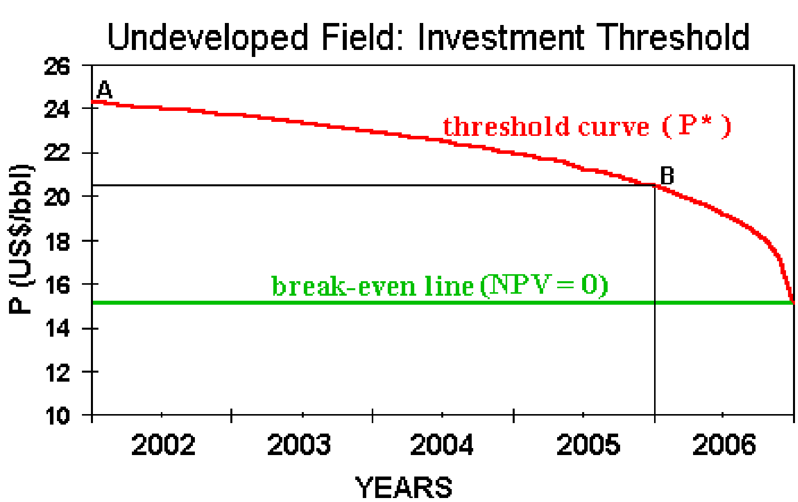
The Undeveloped Oilfield Value: Real Options and NPV

- **Assume that V = q B P, so that we can use chart F x V or F x P**
- **Suppose the development break-even (NPV = 0) occurs at US$15/bbl**



Threshold Curve: The Optimal Decision Rule

- **At or above the threshold line, is optimal the immediate development. Below the line: "wait, learn and see".**
  - **Compare the points "A" and "B" with the previous slide.**



Biodiesel Business Format

- **The biodiesel business format suggested by real options analysis is to enter also in the vegetable raw oil market, by allowing an excess crushing capacity (~small investment) so that we can make biodiesel <u>and</u> vegetable oil to market.**
- **In this way we have two complementary business with a real options *natural hedging* for vegetable oil prices:**
  1. **The <u>biodiesel business</u>, where the vegetable raw oil is *cost* to transesterification (so, a *cheap* raw oil benefits this business); and**
  2. **The <u>vegetable oil to market business</u>, where the vegetable raw oil is *revenue* (so, an *expensive* raw oil benefits this business).**
- **In this format, the vegetable oil is demanded either by biodiesel business or other market (e.g., food).**
  - **It is good for everybody: for the farmers, with grain demand either for biodiesel or for other vegetable oil market; and for Petrobras, capturing the options value from the volatile market.**

The Value of Anticipating Oilfield Development

- **This is based in real life case: a large oil company has a *portfolio of deep-water exploration* assets in a certain area.**
  - **There is more than 80% chances of at least one success.**
    - **The oilfield chance factor (oil existence uncertainty) is modeled with Bernoulli distributions, including correlations when relevant.**
  - **In case of success, we'll start the appraisal phase in order to reveal the reserve volume (B) and quality (q).**
    - **Technical uncertainty over B and q is modeled w/ conditional expectation distributions (conditioning is the information revealed by appraisal wells)**
  - **In addition, oil prices are uncertain and follows a mean-reversion plus jumps process.**
- **The standard investment process is:**
  - **Drill the exploratory prospects (~ one year);**
  - **Drill appraisal wells in case of success (more ~ one year); and**
  - **After completed the appraisal phase, develop the oilfield (if it is deep-in-the-money).**

The Value of Anticipating Oilfield Development

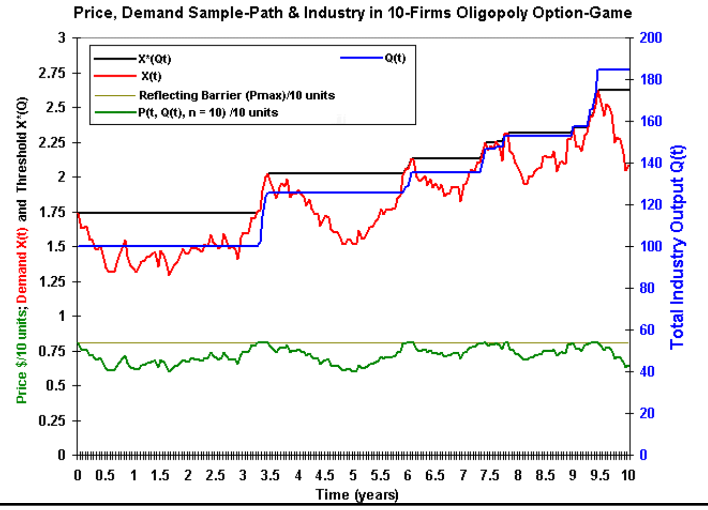
- **After the development decision, the critical path to start oilfield production is the *floating production unit* (FPU).**
- **What if we antecipate the FPU investment, even before the exploratory drilling, in order to antecipate the production (from 1 to 2 years) of the *best* discovered oilfield?**
  - **We have the *option to antecipate production of the best of n risky assets* (exploratory prospects). This is a kind of *rainbow option*.**
    - **We also include one "insurance" asset: one already discovered oilfield (but not *deep-in-the-money*). If all the prospects are dry-holes, we can use the contracted FPU on this oilfield (but FPU will be super-dimensioned).**
  - **We use MC simulation (drilling success; volume and quality; oil prices) in order to quantify the two investment strategy values:**
    - **Traditional strategy (investing in a *taylor-made* FPU but only after the appraisal phase) x antecipating FPU strategy (with a flexible plant).**
  - **For the specific portfolio considered, MC simulation showed that the riskier strategy (earlier FPU investment) is more valuable.**

Oligopoly under Uncertainty

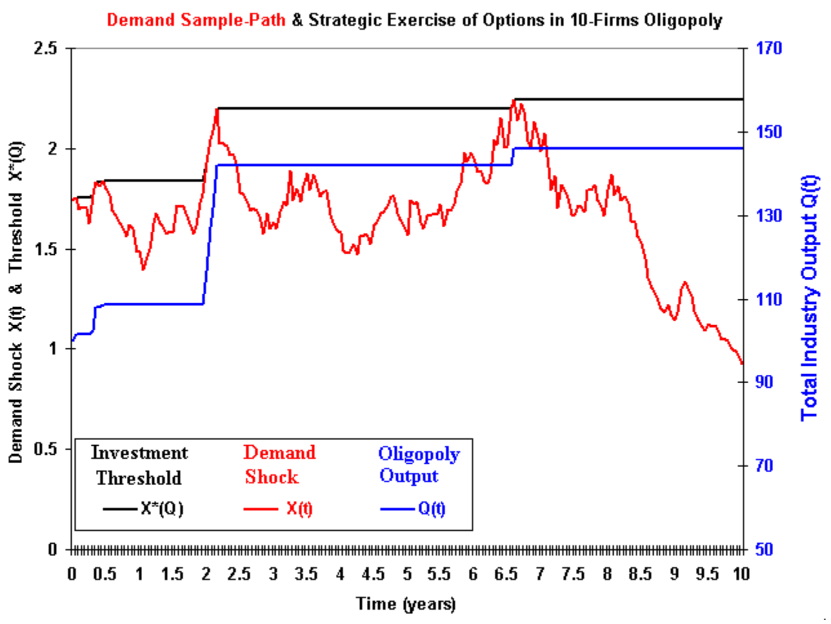
- **Consider an oligopolistic industry with n equal firms**
  - **Each firm holds compound perpetual American call options to expand the production.**
  - **The output price P(t) is given by a demand curve D[X(t), Q(t)]. Demand follows a geometric Brownian motion.**
- **This Grenadier's model has two main contributions to the literature:**
  - **Extension of the Leahy's "principle of optimality of myopic behavior" to oligopoly (myopic firm ignoring the competition makes optimal decision);**
  - **The determination of oligopoly exercise strategies using an "artificial" perfectly competitive industry with a *modified demand function*.**
    - **Both insights simplify the option-game solution because "*the exercise game can be solved as a single agent's optimization problem*" and we can apply the usual real options tools, avoiding complex equilibrium analysis.**
- **We will see some simulations with this model in order to compare the oligopolies with few firms (n = 2) and many firms (n = 10) in term of investment/industry output levels**
  - **We will see the maximum oligopoly price, an *upper reflecting barrier***

Simulation in Real Options Game: Oligopoly

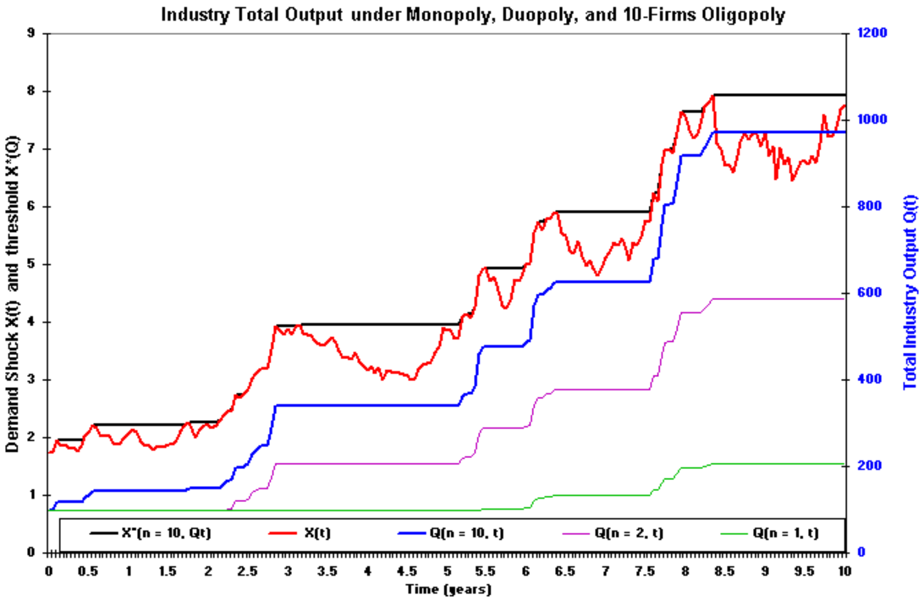
- **Grenadier (2002) is a nice example of using MC to analyze an oligopoly under uncertainty using real options + game theory.**
  - **Simulating demand, we see the effect on industry investment option exercise, production and price (figure below shows one sample-path).**



Oligopoly under Uncertainty



Oligopoly under Uncertainty



Flex Fuel Plant: Input Values

- **In the numerical flex-fuel example, as presented in the chart, were used:**
  - **Volatilities of 25% p.a. (for both, oil and coal)**
  - **Reversion slowness: half-life of 3 years (for both, oil and coal); and**
  - **Interest rate of 6% p.a.**

Return

Quasi-Monte Carlo Numbers: Filling in the Gaps

- **QMC sequence of numbers has the filling in the gaps property: next number in the sequence is placed in the largest gap.**