Oligopoly under Uncertainty - The Grenadier Model

Contents:

1) Introduction

This section is based in S.R. Grenadier's working paper "***Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms***", Stanford University, November 2000. An almost equal version was published at ***Review of Financial Studies*****, vol.15, Summer 2002, pp. 691-721**.

This paper has at least two very important contributions to the option-games literature:

1. Extension of the Leahy's "*Principle of Optimality of Myopic Behavior*" to oligopoly; and
2. The determination of oligopoly exercise strategies using an *"artificial" perfectly competitive industry* with a *modified demand function*.

Both insights simplifly the problems solution because "*the exercise game can be solved as a single agent's optimization problem*" and the usual real options tools in continuous-time. In order to solve the option-game problem it is not necessary more complex techniques to search *fixed-points* from the players best-response correspondences.
It is not by chance that most contributions in option-games come from real options researchers rather than game-theoreticians. Tools like stochastic processes and optimal control are more useful than fixed-point theorems and other related tools. However, another promising way to solve option-games models comes from two game-theoreticians, P.K. Dutta & A. Rustichini (1995, "(s, S) Equilibria in Stochastic Games", *Journal of Economic Theory*, vol.67, 1995, pp.1-39, who prove that the best response map satisfies a *strong monotonicity condition*, which is used to set the existence of Markov-Perfect Nash equilibria.
The third related school that can contribute to option games literature comes from researchers in optimal control, e.g., Basar & Olsder (1995), "Dynamic Non-Cooperative Game Theory" and Dockner & Jorgensen & Van Long & Sorger (2000): "Differential Games in Economics and Management Science", mainly the ***stochastic differential games*** branch. However, the bridge between option games and that branch in optimal control literature remains to be constructed.

In the first insight, the myopic firm (denoted by i) is a firm that when considering the optimal entry in a market, assumes that all the other firms production (denoted by Q_{-i}) will remain constant forever. As Dixit & Pindyck (1994, p.291) mention, "*each firm can make its entry decision ... as if it were the last firm that would enter this industry, and then making the standard option value calculation*" and "*it can be totally myopic in the matter of other firms' entry decisions*". The remarkable property of the *optimality of myopic behavior* was discovered by Leahy (1993, "Investment in Competitive Equilibrium: The Optimality of Myopic Behavior", *Quarterly Journal of Economics*, 1993, pp.1105-1133) and has been used and extended in many ways. See also Baldursson & Karatzas (1997).

This Grenadier's paper is closed related with the Dixit & Pindyck's textbook (mainly chapter 9, section 1; but also chapters 8 and 11). In Dixit & Pindyck (chapter 9), each firm produces only one unit (so, the total industry output is the number of firms), whereas in Grenadier the number of firms is fixed (n) but each firm can add more than one unit of production. I think the Grenadier's way to model oligopoly is more useful and realistic (so, it is an improvement over Dixit & Pindyck), e.g., monopoly, duopoly, and perfect competition are particular cases respectively for n = 1, 2, and infinite. Perhaps for the case of asymmetric firms, the unit production firm approach of Dixit & Pindyck has advantages over the Grenadier's way, because it is only an ordering problem (low-cost firms enter first).
However, in both cases are necessary to assume that *investment is infinitely divisible* (firm i can add an infinitesimal capacity dq by an infinitesimal investment dK), see Grenadier's footnote 13. Although it is more realistic the assumption of discrete-size (lump-sum) additions of capacity by the firms, the approach may be a reasonable approximation in many industries (e.g., new investment is a small fraction of current industry capacity), mainly if the aim is the industry equilibrium study (but less realistic at firm-level decision). This necessary approximation allows the extension of the Leahy's *Principle of Optimality of Myopic Behavior*, which simplifies a lot the problem solution.
However, this assumption is not necessary for the perfectly competitive case of Leahy (see also the wonderful explanation of Dixit & Pindyck, chapter 8, section 2), where the competitive firm analyzes myopically a *lump-sum* investment to enter in the competitive industry.

The second Grenadier's insight permits the application of the important results obtained in perfectly competitive framework into the apparently more complex case of dynamic oligopoly under uncertainty. As example, Grenadier presents an extension of his previous paper on real-estate markets that considers the *time-to-build* feature for a perfectly competitive industry (Grenadier: "Equilibrium with Time-to-Build: A Real Options Approach", in Brennan & Trigeorgis (eds.), *Project Flexibility, Agency, and Competition*, Oxford University Press, 2000). He obtained simple closed-form solutions for the equilibrium investment strategies using this smart artifice.
Other results obtained for perfectly competitive markets could also be easily extended to the oligopoly case. Examples are the results from Lucas & Prescott (1971, "Investment under Uncertainty"), Dixit (1989, "Hysteresis, Import Penetration, and Exchange Rate Pass-Through"), and Dixit (1991, "Irreversible Investment with Price Ceilings"), among others known results.

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2) The Model and Equations

Warning: this page uses the Html 4.0 code for Greek letters. Use a modern browser for better understanding (Internet Explorer 5.5+ or Netscape 6+). If you are looking "σ" as "sigma" (not as "s" or other thing), your browser understand this Html version.

In this model each firm from a n-firms oligopoly holds a sequence of investment opportunities that are like compound perpetual American call options over a production project of capacity addition. The first assumption is that all firms are equal, with technology to produce a specific product. The output is infinitely divisible, and the unity price of this product is P(t). This price changes with the time because the demand D[X(t), Q(t)] evolves as continuous-time stochastic process.
Assume either that the firms are risk-neutral or that the stochastic process X(t) is risk-neutral (that is, the drift is a risk-neutral drift = real drift less the risk-premium).

Initially, as in Grenadier let us consider a more general diffusion process and a more general inverse demand function, given respectively by:

$$\mathbf{dX = \alpha(X)\ dt + \sigma(X)\ dz}$$

$$\mathbf{P(t) = D[X(t), Q(t)]}$$

If $\boldsymbol{\alpha(X) = \alpha \,.\, X}$ and $\boldsymbol{\sigma(X) = \sigma \,.\, X}$, we get the known geometric Brownian motion (GBM).

In the Cournot-Nash perfect equilibrium, strategies are quantities and market clears the price at each state of the demand along the time. Firms choose quantities $q_i{}^*(t)$ (i = 1, 2, ... n) maximizing its payoffs and considering its competitors best response $Q_{-i}{}^*$.

With the simplifying assumption of equal firms, the natural consequence is the choice of symmetric Nash equilibrium, that is, $\mathbf{q_i{}^*(t) = q_j{}^*(t)}$ for all i, j.
We can write the optimal output for n-firms oligopoly symmetric Nash equilibrium as:

$$\mathbf{q_i{}^*(t) = Q^*(t) / n}$$

The exercise price of the option to add a capacity increment of dq is the investment **I . dq**, where I is the unitary investment cost, equal for all firms.
The option to add capacity is exercised by firm i when the demand chock X(t) reaches a threshold level $X^{i}{}^*(q_i, Q_{-i})$.

Grenadier summarizes the equilibrium in his Proposition 1, with a partial differential equation (PDE) and three boundary conditions. The first and second boundary conditions are the value-matching and smooth-pasting conditions, which are very known in continuous-time real options framework. However, the third condition is the strategic one, requiring that each firm i is maximizing its value $\mathbf{V^i(X, q_i, Q_{-i})}$ given the competitors' strategies (thresholds).
The third condition is a value-matching at the competitors' threshold $X^{-i}(q_i, Q_{-i})^*$, which is equal to $X^{i}(q_i, Q_{-i})^*$ due to the symmetric equilibrium.
The third condition is like a fixed-point search of best response maps. However, this condition will not be necessary with the Grenadier's Proposition 2, extending the myopic optimality concept to oligopolies. Proposition 2 assumes that *investment is infinitely divisible* (see discussion in the introduction) and tells that the myopic firm threshold is equal to the firm's strategic (Cournot-Nash perfect equilibrium) threshold. Proposition 3 will set the main equilibrium parameters with only two boundary conditions.

Denote the value of myopic firm by $M^i(X, q_i, Q_{-i})$. Let us to work with the ***value of a myopic firm's marginal output*** $m^i(X, q_i, Q_{-i})$ defined by:

$$\mathbf{m^i(X, q_i, Q_{-i}) = \partial M^i(X, q_i, Q_{-i}) / \partial q_i}$$

Given the symmetry, we can write $X^{i}(q_i, Q_{-i})^* = X^*(Q)$ because $\mathbf{q_i = Q/n}$ and $\mathbf{Q_{-i} = (n - 1) \,.\, Q / n}$.
Proposition 3 establish the symmetric Nash equilibrium: *each firm will exercise its investment option*

whenever X(t) rises to the trigger X(Q). Let m(X, Q) denote the value of a myopic firm's marginal investment*. The following PDE and two boundary conditions determine both X*(Q) and m(X, Q):

$$0.5\ \sigma(X)^2\ m_{XX} + \alpha(X)\ m_X - r\ .\ m + D(X, Q) + (Q / n)\ D_Q(X, Q) = 0$$

Subject to:

$$m[X^*(Q), Q] = I \text{ ... (value-matching condition at } X^*(Q))$$

$$\partial m[X^*(Q), Q] / \partial X = 0 \text{ ... (smooth-pasting condition)}$$

Where the subscripts in the PDE denote partial derivatives. The red terms of the PDE comprise the non-homogeneous part of the PDE. Dixit & Pindyck's readers easily identify this part as the "cash-flow" terms. This red part will play a very special role in Grenadier's paper, because it is the ***modified demand function*** mentioned early in the introduction.
The black terms of the PDE comprise the homogeneous part of the PDE. It is very known in real options literature.
The two "real options" boundary conditions at the common threshold level X*(Q) are sufficient for the optimal strategic exercise of the option due the Proposition 2 (the myopic firm threshold is equal to the firm's strategic threshold).

Grenadier's section 5 shows that, besides the monopoly and perfectly competitive industry cases, it is also possible to solve the oligopoly case as a single agent optimization problem. The procedure is just to pretend that the industry is perfectly competitive, maximizing a "fictitious" objective function.
This "fictitious" objective function uses an ***"artificial" demand function*** defined by:

$$D'(X, Q) = D(X, Q) + (Q / n)\ D_Q(X, Q)$$

As mentioned in the introduction, this is a very important result because permits the extension of known (or easier obtained) results in perfectly competitive setting to the oligopoly case.
In section 6, Grenadier shows an equilibrium with time-to-buid example of this extension. Here I present below the example of his section 3 (with some further simulations don't showed in the paper).

Consider the more specific diffusion process - geometric Brownian motion (GBM) - and also a more specific inverse demand function - a multiplicative chock constant-elasticity demand curve - given respectively by:

$$dX = \alpha\ X\ dt + \sigma\ X\ dz$$

$$P(t) = X(t)\ .\ Q(t)^{-1/\gamma}$$

Where $\gamma > 1/n$ to ensure that marginal profits are increasing in X. Assume also that the risk-free discount rate is strictly higher than the (risk-neutral) drift α.
The optimal threshold X*(Q) is given by:

$$X^*(Q) = v_n\ .\ Q^{1/\gamma}$$

Where $\mathbf{v_n}$ is an ***upper reflecting barrier***, that is, the **maximum price** that the product can reach in the oligopolistic market. When prices reaches this level, firms add capacity in a quantity so that the price is reflected-down due the additional supply.
For this multiplicative demand chock, while X(t) follows a (unrestraint) GBM, the price P(t) follows a GBM with upper reflecting barrier v_n, given by:

$$v_n = \left(\frac{\beta_1}{\beta_1 - 1}\right)\left(\frac{n\,\gamma}{n\gamma - 1}\right)(r - \alpha)\,.\,I$$

Where $\beta_1 > 1$ is the known positive root of the quadratic equation: $0.5\,\sigma^2\,\beta(\beta - 1) + \alpha\,.\,\beta - r = 0$

Note that the threshold X*(Q) is decreasing with n (number of firms in the oligopoly).

The addition of capacity dQ (= n dq) when X(t) > X*(Q), which costs I. dQ, is larger as larger is the difference X(t) - X*(Q) (in order to keep the prices not above v_n).
In other words, if X(t) > X*(Q) then $\mathbf{Q(t) = (X(t) / v_n)^{\gamma}}$

In the next section is presented some numerical simulations of this example, with some charts and a spreadsheet that will help to understand the model.
Before the simulations, a last issue: What is the ***option premium*** when exercising this strategic option in the n-firms oligopoly? Grenadier defines this option premium as the NPV at X* per unit of investment I, **OP(n)** given by:

$$\mathbf{OP(n) = 1 / [(n\,.\,\gamma) - 1]}$$

So, when n tends to infinite the OP(n) tends to zero (zero NPV for the perfectly competitive case, as in Dixit & Pindyck, chapter 8). For n finite the NPV is positive but as small as large is the number of firms n.

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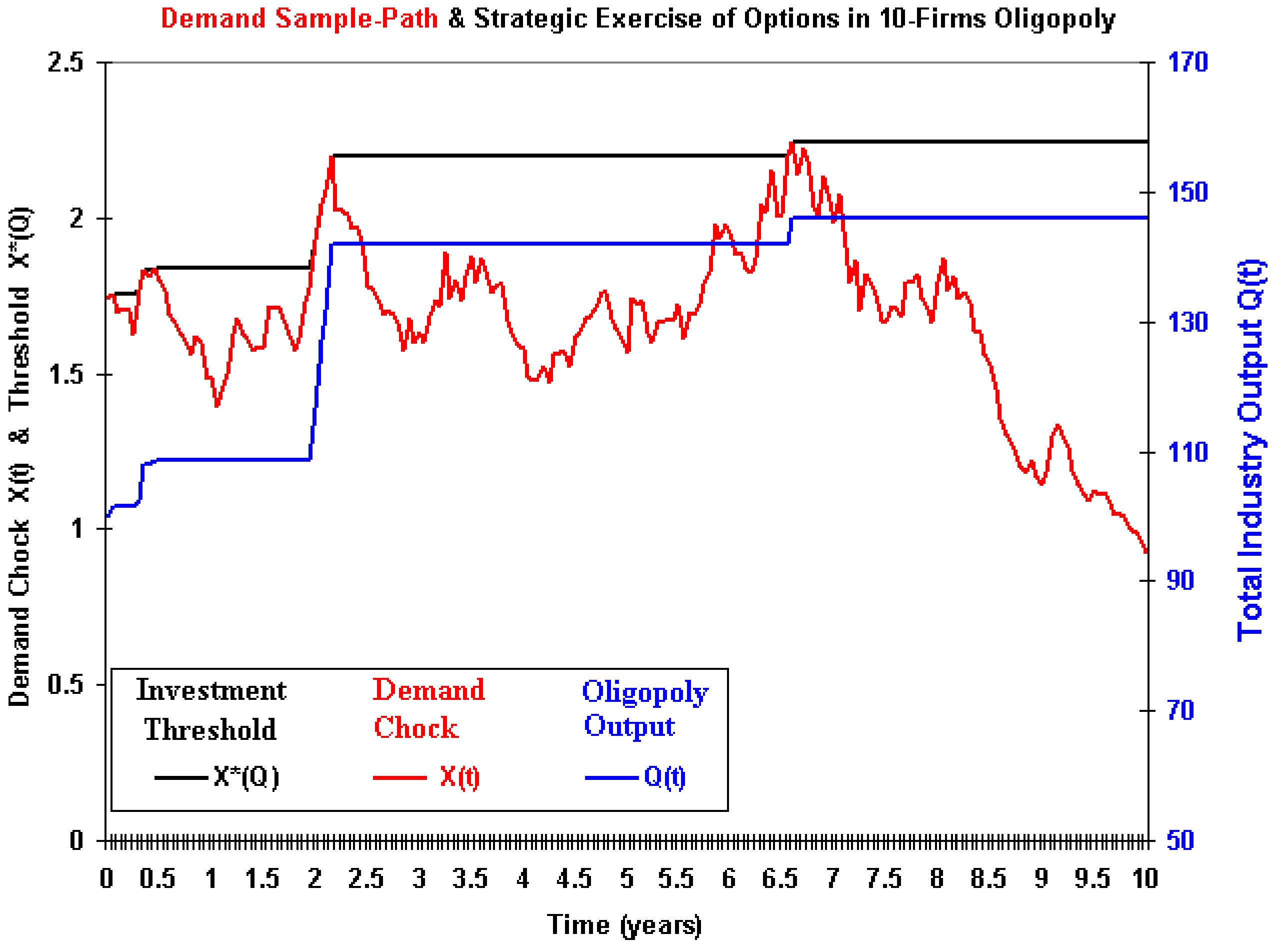
3) Oligopoly Simulation and the Spreadsheet Downloading

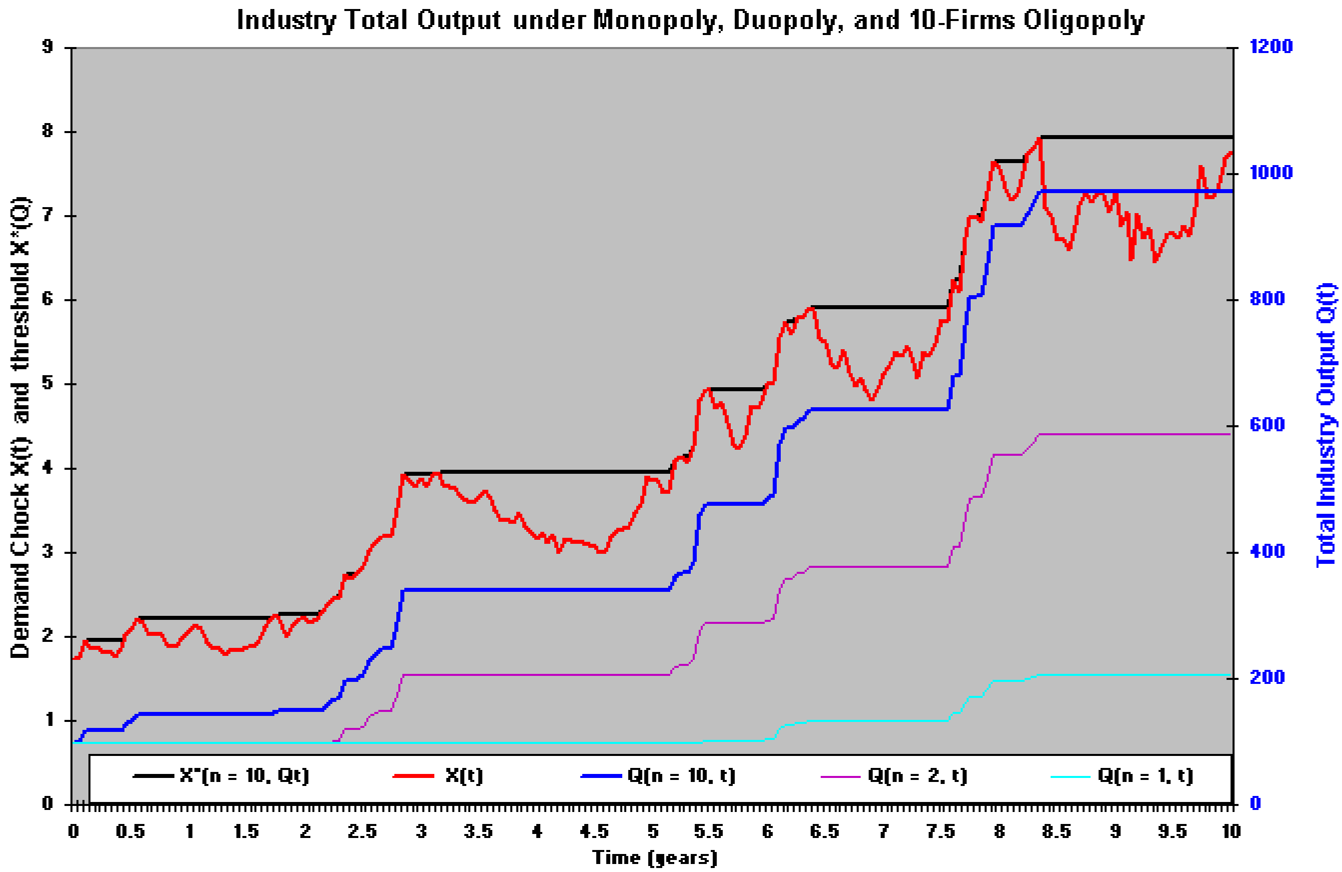
In this section, I use the same numerical values used in Grenadier's paper, section 3 for his figure 1.
The values are: $\alpha = 0.02$; $r = 0.05$; $\sigma = 0.175$; $\gamma = 1.5$; n = 10 firms; I = 1; Q(0) = 100; and X(0) = 1.74.
All the figures below and the base-case in the spreadsheet assumes these values, except where indicated.

An interesting and practical feature of the ***principle of optimality of myopic threshold*** is that we can use **Monte Carlo simulation** in order to solve the model. **It is not necessary to work backwards** because we know the ("myopic") threshold level X*(Q(t)) in advance. So, if this threshold is triggered by the simulated sample-path of the demand X(t), new capacity is added by the oligopolistic firms and its is easy to study the aggregate behavior of the industry in the long-run (like the industry output Q(t), the investment along the years, the prices evolution, etc.) and many properties of the strategic exercise of options in an oligopoly.
The figure below shows these features for a certain demand sample-path X(t) in 10-firms oligopoly.



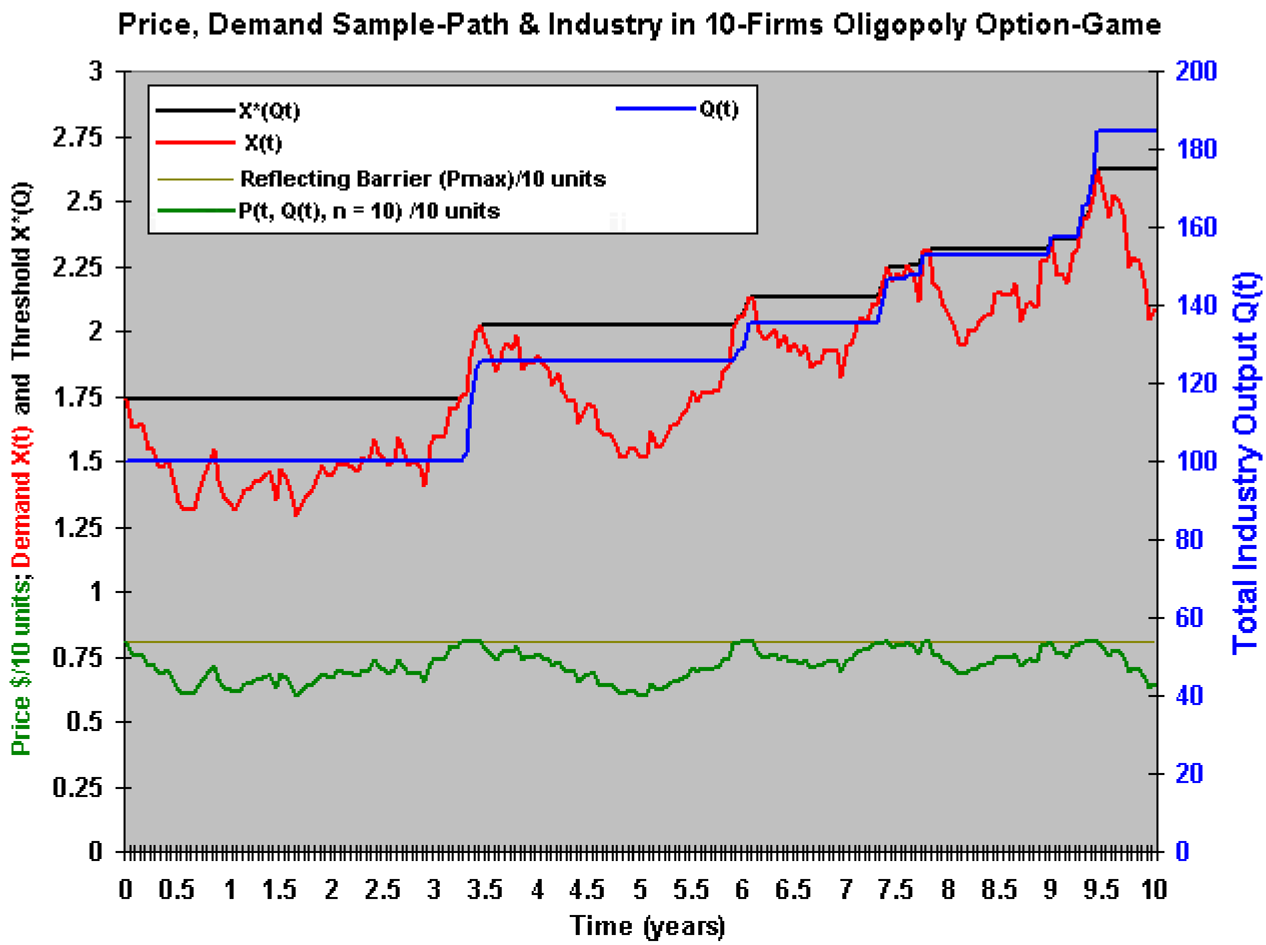
In this model, the addition of capacity (investment) from the firms is proportional to the difference between the the demand chock X(t) and the threshold level X*(Q(t)), if positive, and no investment in case of the demand below the threshold X*.
In Grenadier's model the firms are equals, so that in 10-firms oligopoly case each firm adds 1/10 of the capacity addition Q(t) - Q(t - dt) in case of a positive chock at t if X(t) > X*(Q(t - dt)).

The figure below shows for one demand evolution sample-path, that the industry total output Q(t) is much more for the 10-firms oligopoly (n = 10) case than for duopoly (n = 2), which also presents a higher industry output than the monopoly case (n = 1).



The figure above shows that, for the same demand evolution, after 10 years the oligopoly with 10 firms produce together near 1000 units, whereas the duopoly and monopoly produce near 600 and 200 units, respectively.

The figure below shows the evolution of the prices, considering the demand chock and the oligopoly addition of capacity. Note that there is an upper reflecting barrier at $ 0.8081/10 units, so that when exists a positive demand reaching this reflecting barrier, the oligopoly addition of capacity is sufficiently high to prices stay at this level (if demand remains rising) or to be reflected-down. The figure below shows this feature for a certain demand sample-path.



Download the Excel spreadsheet **simulation-oligopoly_equilibrium.xls** (file with 291 KB), and see 4 charts - including the 3 charts from the above figures. Press **F9** in order to see new sample-paths with the consequents results in terms of thresholds, industry production, prices, etc.
The *non-registered version* can be freely downloaded below and used for educational purposes:

- **Download** Excel file named **simulation-oligopoly_equilibrium.xls** with 291 KB

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