

# Monte Carlo Simulation of Stochastic Processes

Last update: January 10th, 2004.

In this section is presented the steps to perform the simulation of the main stochastic processes used in real options applications, that is the *Geometric Brownian Motion*, the *Mean Reversion Process* and the combined process of *Mean-Reversion with Jumps*.

For all cases I present both simulations the *risk-neutral* and the *real* one. Risk-neutral simulations are used for derivatives pricing, whereas real simulations are used for value at risk, hedging and, for real options applications, is useful for example to find the probability of exercise the option and the expected time to wait before the option exercise (see the new [Timing spreadsheet](#)).

See the [FAQ 4](#) for a discussion of risk-neutral versus real simulations.

This section includes the topics:

◆ [Monte Carlo Simulation of Geometric Brownian Motion](#)

◆ [Monte Carlo Simulation of Mean Reversion \(Model 1\)](#)

◆ Complement: [Discretization Accuracy of the Mean-Reversion Stochastic Process](#). **NEW!**  
... **Download** an Excel [spreadsheet that simulates this mean-reversion model](#) and discusses the discretization accuracy. **NEW!**

◆ [Monte Carlo Simulation of Mean Reversion \(Model 2\)](#).

◆ [A More Practical Approach to Mean Reversion Model 2](#). **NEW!** November 2003 insertion .

◆ [Monte Carlo Simulation of Mean Reversion with Jumps](#)

... **Download** a [spreadsheet simulating the mean-reversion + jumps](#) sample paths. **NEW!**

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## Monte Carlo Simulation of Geometric Brownian Motion

Consider that the price  $P$  of a commodity follows a Geometric Brownian Motion, which is given by the following stochastic equation:

$$dP = \alpha P dt + \sigma P dz$$

Where  $dz =$  Wiener increment  $= \varepsilon dt^{0.5}$ ,  $\varepsilon$  is the standard normal distribution;  $\alpha$  is the drift (or capital gain rate); and  $\sigma$  is the volatility of  $P$ .

By using the equation of *total investment return*  $\mu = \alpha + \delta$ , where  $\mu$  is also the **risk-adjusted**

**discount rate** for P; and  $\delta$  is the dividend yield (or convenience yield in case of commodities). We can rewrite the stochastic equation as:

$$dP = (\mu - \delta) P dt + \sigma P dz$$

For the risk-neutral version of this equation, just replace the risk-adjusted discount rate  $\mu$  by the *risk-free interest rate*  $r$  to obtain the *risk-neutral stochastic equation*:

$$dP = (r - \delta) P dt + \sigma P dz$$

Using a logarithm transformation and applying the Itô's Lemma, we can reach the equations for the prices simulation in both formats, real and risk-neutral.

For more technical details, see the page on [Geometric Brownian Motion](#), from the Stochastic Processes section.

The *real simulation of a GBM* uses the real drift  $\alpha$ . The price  $P_t$  at the future instant  $t$  is given by:

$$P_t = P_0 \exp\{ (\alpha - 0.5 \sigma^2) \Delta t + \sigma N(0, 1) \sqrt{\Delta t} \}$$

The simulation of the real prices using the above equation is done by sampling the standard Normal distribution  $N(0, 1)$  and obtaining the correspondent values for  $P_t$ . These values of  $P$  can be used to calculate the (real) values of the project  $V$  by means of some equation  $V(P)$  or combined even with other uncertain variables in order to obtain the real value of  $V$ .

Remember, with the real drift simulation of the underlying asset ( $P$ ), the required discount rate is the risk-adjusted one  $\mu$ . Real simulations are used mainly for the calculus of the probability of option exercise and for the expected time of option exercise (given conditional to have occur at least one exercise in the simulation), see the new [Timing](#) spreadsheet. In financial applications, the real simulation is used for VaR (value at Risk) estimation.

However, for derivatives in general is required other type of simulation, the risk-neutral one, because the risk-adjusted discount rate for the derivative  $F(P)$  in general is not the same  $\mu$  used for  $P$ .

The *risk-neutral simulation of a GBM* uses the risk-neutral drift  $\alpha' = r - \delta$ . The price at  $t$  is:

$$P_t = P_0 \exp\{ (r - \delta - 0.5 \sigma^2) \Delta t + \sigma N(0, 1) \sqrt{\Delta t} \}$$

The simulation of the *risk-neutral prices* using the above equation is performed by sampling the standard Normal distribution  $N(0, 1)$  and obtaining the correspondent values for  $P_t$ . The risk-neutral simulation is used mainly for the derivatives valuation in complete markets. For incomplete markets, is possible use either the risk-neutral one (but need to select a martingale measure from several or to estimate the risk-premium in this market) or the real simulation one with an exogenous risk-adjusted discount rate (dynamic programming), see [FAQ 5](#).

However, for both the estimation of the probability of option exercise and the expected time of option exercise, the correct is the real simulation and not the risk-neutral one.

One important feature of the above discrete-time equations is that the discretization from the

continuous-time model is exact. In other words, you don't need to use small time increments  $\Delta t$  in order to get a good approximation. You can use any  $\Delta t$  that the simulation equation is valid (check out this affirmative by evaluating an [European call option](#) with time to maturity of one year, for example, by risk-neutral simulation of the underlying asset only at the expiration, and conferring with the exact Black-Scholes formula result).

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## Monte Carlo Simulation of Mean Reversion (Model 1)

Initially consider the following *Arithmetic Ornstein-Uhlenbeck* process for a stochastic variable  $x(t)$ :

$$dx = \eta (\bar{x} - x) dt + \sigma dz$$

This means that there is a reversion force over the variable  $x$  pulling towards an *equilibrium level*  $\bar{x}$ , like a spring force.

The *velocity of the reversion* process is given by the parameter  $\eta$ .

This stochastic differential equation is *explicitly solvable* (see Kloeden & Platen, 1992, topic 4.4, eq.4.2) and has the following solution in terms of stochastic integral (Itô's integral):

$$x(T) = x(0) e^{-\eta T} + (1 - e^{-\eta T}) \bar{x} + \sigma e^{-\eta T} \int_0^T e^{\eta t} dz(t)$$

The variable  $x(T)$  has *Normal distribution* with the following expressions for the mean and variance (see for example Dixit & Pindyck, 1994, chapter 3):

$$E[x(T)] = x(0) e^{-\eta T} + \bar{x} (1 - e^{-\eta T})$$

$$\text{Var}[x(T)] = (1 - e^{-2\eta T}) \cdot \frac{\sigma^2}{2\eta}$$

In the expected value equation note that the mean is just a weighted average between the initial level  $x(0)$  and the long-run level  $\bar{x}$  (the weights sum one and are functions of the time and the reversion speed).

The variance increases with the time, but also converges to  $\sigma^2/(2\eta)$  as the time goes to infinite. See the textbook of Oksendal ("Stochastic Differential Equations - An Introduction with Applications") for details on Itô's stochastic integration, and the exercise (with answer) for the mean-reversion process.

There is the following relation between the mean-reversion speed ( $\eta$ ) and the *half-life* ( $H$ ) of the process:  $H = \ln(2)/\eta$  ([click here for the proof](#)). Half-life is the time that is expected time for the stochastic variable  $x$  to reach the half of way toward the equilibrium level  $\bar{x}$ .

In order to perform the simulation is necessary to get the discrete-time equation for this process. The correct discrete-time format for the continuous-time process of mean-reversion is the stationary first-order autoregressive process, AR(1), see Dixit & Pindyck, 1994, p.76 (for  $\Delta t = 1$ , the equation below is more general). So the sample path *simulation equation* for  $x(t)$  is performed by using the *exact* (valid for large  $\Delta t$ ) discrete-time expression:

$$\mathbf{x}_t = \mathbf{x}_{t-1} e^{-\eta \Delta t} + \bar{\mathbf{x}} (1 - e^{-\eta \Delta t}) + \sigma \sqrt{(1 - \exp(-2 \eta \Delta t)) / (2 \eta)} \mathbf{N}(0,1)$$

Schwartz (1997, footnote 15) also recognizes the existence of an "exact transition equation", but in the last term he uses an approximation to the variance instead the exact one presented in Dixit & Pindyck.

Note that the equation above can be viewed as the sum of the expected value equation with a random term (appear the standard Normal distribution) with zero mean.

The discretization alternatives with either the first-order Euler's approximation or the Milstein's approximation formats introduce discretization errors into the simulation and have higher computational cost because need small  $\Delta t$ .

The AR(1) discretization presented above is exact in the sense that we can use large  $\Delta t$  without any problem in terms of simulation accuracy. See more details in the specific topic below.

Note that  $x(t)$  has Normal distribution and can assume even negative values. In many applications we want that the stochastic variable assume only positive values, for example in the case of price of commodities  $P$ . How build a good mean-reversion model for  $P$ ? What is the relation between  $P$  and our variable  $x(t)$ ? What is the relation between the long-run level  $\bar{\mathbf{x}}$  and a *long-run equilibrium level for a commodity price  $\bar{P}$* ?

In this **Model 1** (the Model 2 is different at this point), we have two important simple concepts. First, the model identifies the long-run equilibrium level for a commodity price as a very important reference, so that assume that the commodity price  $P$  follows a mean-reversion toward an equilibrium level  $\bar{P}$  given by the equation:

$$\bar{\mathbf{x}} = \ln(\bar{P})$$

Or the long-run equilibrium level for the commodity is simply  $\bar{P} = \exp(\bar{\mathbf{x}})$ , which is constant here.

The second concept from this model 1 is to set the prices in a way that the simulated mean become  $E[P(T)] = \exp\{E[x(T)]\}$ , that is, the relation between the variables  $x$  and  $P$  is such that in the simulation we get the following expected value for the commodity price at the instant  $T$ :

$$E[P(T)] = \exp\{ \mathbf{x}(0) e^{-\eta T} + \bar{\mathbf{x}} (1 - e^{-\eta T}) \}$$

The direct process  $P(t) = \exp[x(t)]$  doesn't work here because the exponential of a Normal distribution adds the half of the variance in the log-normal distribution mean. In order to reach the goal ( $E[P(T)] = \exp\{E[x(T)]\}$ ), the half of the variance is compensated using the equation:

$$P(t) = \exp\{ \mathbf{x}(t) - 0.5 \text{Var}[\mathbf{x}(t)] \}$$

Where  $\text{Var}[x(t)]$  is a deterministic function of time given before.

With the equations above is easy to simulate the **real** sample paths for the commodity prices  $P$  following a mean-reversion process. Just simulate  $x(t)$  by sampling the Normal distribution and using the equation for  $x(t)$  given before. Calculate  $\text{Var}[x(t)]$  and use the equation above in order to calculate the (simulated) value for  $P(t)$ . Make this along the path (along the time for every discrete instant  $t$ ) and for  $n$  simulated paths.

By combining these three equations, you can simulate the real process for  $P(t)$  directly with the equation:

$$P(t) = \exp \left\{ \left[ \ln[P(t-1)] \exp[-\eta \Delta t] + \left[ \ln(\bar{P}) (1 - \exp[-\eta \Delta t]) \right] - \left[ (1 - \exp[-2 \eta \Delta t]) \frac{\sigma^2}{4 \eta} \right] + \sigma \sqrt{\frac{1 - \exp[-\eta \Delta t]}{2 \eta}} N(0, 1) \right] \right\}$$

The interpretation for the 4 terms inside the first  $\exp\{\cdot\}$  is: (a) the first and second terms are drift terms which weight the initial value and the long-run mean (note that the weights sum 1); (b) the third term is the convex adjustment (or Jensen's inequality adjustment to the drift); and (c) the fourth term is the stochastic term (with the standard Normal), which we sample in the Monte Carlo simulation.

Note also that the volatility parameter enters only in the third and fourth terms, not in the drift, as tell our intuition.

The framework above permits the *simulation of the real process*. Let us see how to perform the simulation of the *risk-neutral* mean-reverting process.

Remember that the risk neutral format permits to value derivatives and requires the use of the risk-free interest rate as the correct discount rate. As before, the process is risk neutralized by changing the drift. In this risk-neutral format, the drift  $\alpha$  of the process is replaced by  $r - \delta$ , where  $\delta$  is the dividend yield.

For the *mean-reversion case*, the **real drift** is  $\alpha = \eta(\bar{x} - x)$ , and the dividend yield is not constant, it is function of  $x$ :

$$\delta = \mu - \alpha = \mu - \eta(\bar{x} - x)$$

Where  $\mu$  is the risk-adjusted discount rate for the underlying asset  $x$  (remember the total return is  $\mu = \alpha + \delta$ ).

With this expression we get the *risk-neutral drift for the mean-reversion process*:

$$r - \delta = r - \mu + \eta(\bar{x} - x) = \eta(\bar{x} - x) - (\mu - r) = \eta\{\bar{x} - ((\mu - r)/\eta) - x\}$$

Note that  $\mu - r$  is the *risk-premium*. The comparison between both drifts indicates that the passage from the real process to the risk-neutral one, can be viewed as subtracting the *normalized risk-premium*  $((\mu - r)/\eta)$  from the long run mean level  $\bar{x}(= \ln \bar{P})$ . In other words, in the risk-neutral process the prices revert toward a level that is lower than the real long-run level, and the subtracting term is a kind of normalized risk-premium.

By substituting the risk-neutral drift in the mean-reversion stochastic process, we can reach a

slight different equation for the simulation of  $x(t)$  and so for  $P(t)$ .  
Hence, the continuous-time *risk-neutral* stochastic equation for  $x$  is given by:

$$dx = \eta \left( \left[ \bar{x} - \frac{\mu - r}{\eta} \right] - x \right) dt + \sigma dz$$

In the *risk-neutral* format, the process  $x(t)$  is simulated using the exact (valid for large  $\Delta t$ ) discrete-time expression:

$$x_t = x_{t-1} e^{-\eta \Delta t} + \left[ \bar{x} - \frac{(\mu - r)}{\eta} \right] (1 - e^{-\eta \Delta t}) + \sigma \sqrt{(1 - \exp(-2 \eta \Delta t)) / (2 \eta)} N(0, 1)$$

The red term is all the difference compared with the real simulation format (compare with the equation for  $x(t)$  presented before).

The process simulation is simple as in the real process case. Calculate  $x(t)$  with the equation above and sampling the standard Normal distribution  $N(0, 1)$ . With the simulated values of  $x(t)$ , use equation of price  $P(t) = \exp\{x(t) - 0.5 \text{Var}[x(t)]\}$  together the equation for the variance of  $x(t)$  in order to get  $P(t)$ .

By combining these three equations, you can simulate the risk-neutral process for  $P(t)$  directly with the equation:

$$P(t) = \exp \left\{ \left[ \ln[P(t-1)] \exp[-\eta \Delta t] + \left[ \ln(\bar{P}) - \frac{(\mu - r)}{\eta} \right] (1 - \exp[-\eta \Delta t]) \right] - \left[ (1 - \exp[-2 \eta \Delta t]) \frac{\sigma^2}{4 \eta} \right] + \sigma \sqrt{\frac{1 - \exp[-\eta \Delta t]}{2 \eta}} N(0, 1) \right\}$$

The interpretation of the 4 terms inside the first  $\exp\{\cdot\}$  is analogous to the interpretation mentioned for the real simulation (two drift terms, a convex adjustment term, and the stochastic term) but with the drift terms weighting the initial value and the *risk-neutral* long-run mean instead the *real* long-run mean.

The spreadsheet below (non-protected) simulate both real and risk-neutral sample-paths of the mean-reversion process described above (Model 1). **NEW!**

◆ [Download the Excel spreadsheet simulation reversion-modell.xls, with 45 KB](#)

## Discretization Accuracy of Mean-Reversion Stochastic Process

This discretization issue deserves a special discussion. The above discrete-time equations for the Arithmetic Ornstein-Uhlenbeck  $x(t)$  permits an exact discretization in the sense that the accuracy doesn't decrease if a larger time-step ( $\Delta t$ ) is used.

Because the error from the discretization is eliminated using the above exact discretization equation, the only error that remains in the simulation is the error from the random numbers.

This topic discusses the motivation to use a better discretization when available, including reference to a list of stochastic processes that permit an exact discretization. This topic also presents an Excel spreadsheet - available to download, that shows in practice the superiority of the exact discretization over the popular Euler's approximation.

The use of large  $\Delta t$  is much more important in real options (long-term options) than in financial options - although it is useful also for European financial options. Perhaps this explain that most books ignore the exact simulation equation for mean-reversion, insisting to teach only Euler and Milstein approximations even when exact solutions are available.

One exception is the textbook of **Kloeden & Platen (1992)** that in the topic 4.4 list "*Some explicitly solvable SDEs*". The Arithmetic Ornstein-Uhlenbeck is their equation 4.2 (for the geometric Brownian motion, see eq. 4.6 from this textbook). Kloeden & Platen use the explicitly solvable SDEs (solution doesn't depend on small  $\Delta t$ ) to check the accuracy of numerical methods that depends on  $\Delta t$  (like Euler approximations). See pp.308-310 for the geometric Brownian example.

The **spreadsheet** below **simulate this mean-reversion stochastic process, Model 1**, using *three different discretization methods* in order to show that the exact method presented above is the most accurate one.

The spreadsheet also plots the **histogram chart** (the theoretical is a log-normal distribution) for the simulated values of the mean-reverting commodity price at a specified time T and for the other specified parameters set by the user.

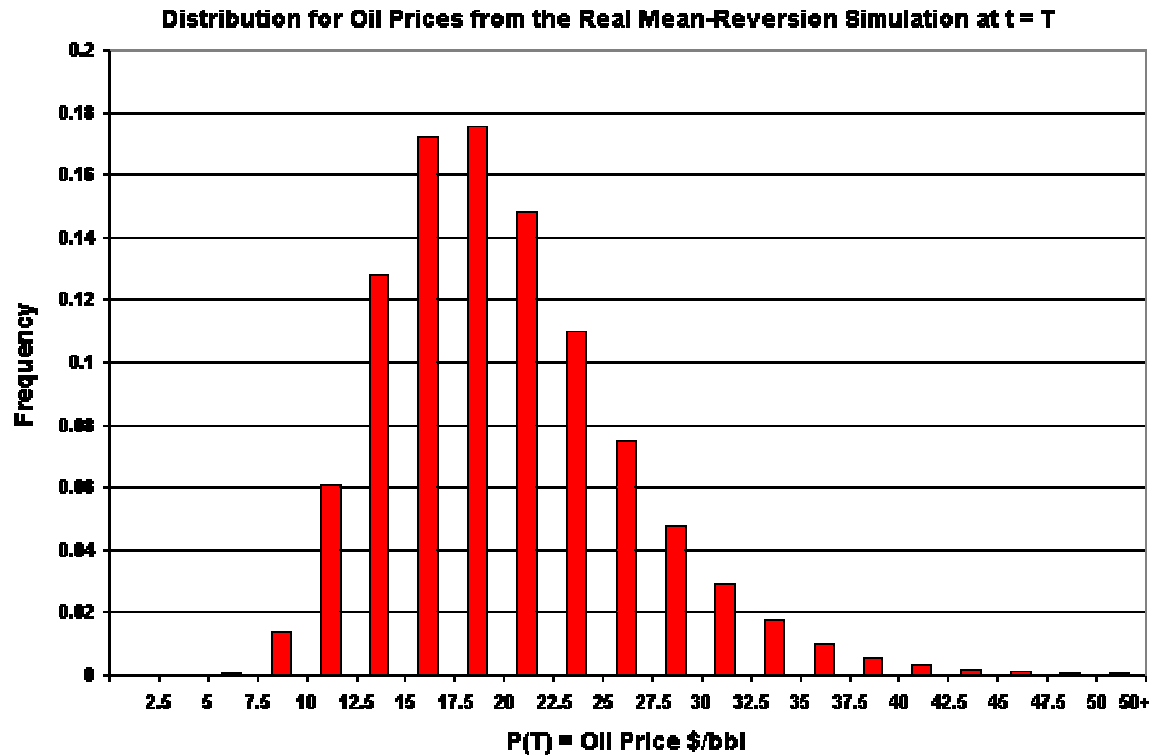
◆ **Download the Excel spreadsheet [reversion-simulation\\_accuracy-vba.xls](#), with 372 KB**

The discretization presented before is compared with the popular first order Euler approximation. In short, it uses a discrete time-step  $\Delta t$  substituting directly the differential of time  $dt$ . The same is done for the increment  $dx$ . The discrete-time version of the Arithmetic Ornstein-Uhlenbeck using the first order Euler's approximation is given by the equation below (see for example the book of Clewlow & Strickland, 2000, p.110, eqs. 7.3 and 7.4):

$$\Delta x = \eta (\bar{x} - x) \Delta t + \sigma \Delta z$$

As commented before, this equation is less accurate and the accuracy depends of a small time-step  $\Delta t$ . So, with Euler approximation, in addition to the simulation error (from the random numbers), there is the discretization error.

The figure below, a histogram for the oil prices 10 years ahead, was drawn using the spreadsheet above.



## Monte Carlo Simulation of Mean Reversion (Model 2)

The Model 2 is similar to the model 1 in many aspects, but has important differences. This model is known as "Schwartz Model 1", from his famous paper of 1997 in *Journal of Finance* (although Schwartz prefers other mean reversion models, e.g., *two and three factors* models).

The model uses the same equation for  $x(t)$  indicated above for the Model 1. The differences are:

- The relation between  $x(t)$  and  $P(t)$  is simpler,  $P(t) = \exp[x(t)]$ ;
- The relation between  $\bar{x}$  and  $\bar{P}$  is much more complicated than the case presented before; and
- The simulated mean for the prices is *not*  $E[P(T)] = \exp\{E[x(T)]\}$ .

In other words there are advantages and disadvantages for this model when compared with the mean-reversion case analyzed previously (Model 1).

The other advantage of the Schwartz's model is that the application of the Itô's Lemma, in order to get the continuous time stochastic differential equation for the  $P(t)$  process, is easier than model 1. However, for simulation purposes this step is not necessary (only for the PDE approach is necessary). The correspondent continuous-time stochastic differential equation for the commodity prices is:



$$dP = \eta(\ln \bar{P} - \ln P) P dt + \sigma P dz$$

This is the Schwartz's eq.1 with other notation. The relation between  $\bar{x}$  and  $\bar{P}$  is given by the equation (see the eq. 3 in Schwartz, 1997):

$$\bar{x} = \ln \bar{P} - (\sigma^2 / (2\eta))$$

Hence the expected long-run equilibrium price (real process) depends on both the volatility and the reversion speed.

The *real simulation* of this model is given by the following equation that again uses an exact discretization that allows large  $\Delta t$  (correcting an old version of this website):

$$P_t = \exp\{ [\ln(P_{t-1}) (\exp(-\eta \Delta t))] + [(\ln \bar{P} - (\sigma^2 / (2\eta))) (1 - \exp(-\eta \Delta t))] + [\sigma (\text{SQRT}((1 - \exp(-2\eta \Delta t)) / (2\eta))) N(0,1)] \}$$

Where the SQRT(.) is the square root operator. The simulation again is very simple, you get the sample paths of  $P(t)$  by sampling the Normal distribution. and using the equation above.

The *expected value for  $P(t)$*  in the simulation is given by taking the expectations (including the convexity adjustment) in the above equation:

$$E[P_t] = \exp\{ [\ln(P_{t-1}) (\exp(-\eta \Delta t))] + [(\ln \bar{P} - (\sigma^2 / (2\eta))) (1 - \exp(-\eta \Delta t))] + [(\sigma^2 / (4\eta)) (1 - \exp(-2\eta \Delta t))] \}$$

In the long-run (large  $t$ ), the *real simulation converge* to the following expected long-run level:

$$F = E[P(\infty)] = \text{Expected Long-Run Futures Price} = \exp[\bar{x} + (\sigma^2 / (4\eta))] = \exp[\ln \bar{P} - (\sigma^2 / (4\eta))] = \bar{P} \exp[-\sigma^2 / (4\eta)]$$

For the *risk-neutral simulation*, just subtract from  $\bar{x}$  (here =  $\ln \bar{P} - \sigma^2 / [2\eta]$ ) the *normalized risk-premium*  $(\mu - r) / \eta$ , that is, the risk-neutral equation is given by:

$$P_t = \exp\{ [\ln(P_{t-1}) (\exp(-\eta \Delta t))] + [(\ln \bar{P} - (\sigma^2 / (2\eta) - ((\mu - r) / \eta)) (1 - \exp(-\eta \Delta t))] + [\sigma (\text{SQRT}((1 - \exp(-2\eta \Delta t)) / (2\eta))) N(0,1)] \}$$

If you take expectation of the above equation you get the Schwartz's eq.7 for the expected value of the futures prices in the risk-neutral format (or "under martingale measure", as Schwartz prefers) because the random term (the last one inside the exponential operator) has zero mean (from standard Normal distribution) and considering the convexity adjustment.

On similar way, the expected commodity price in the *risk-neutral* simulation converges in the long-run (very large  $t$ ) to the following equation (compare with Schwartz eq. 38):

$$\text{Expected Long-Run Risk-Neutral Futures Price} = \exp[\bar{x} - ((\mu - r) / \eta) + (\sigma^2 / (4\eta))]$$

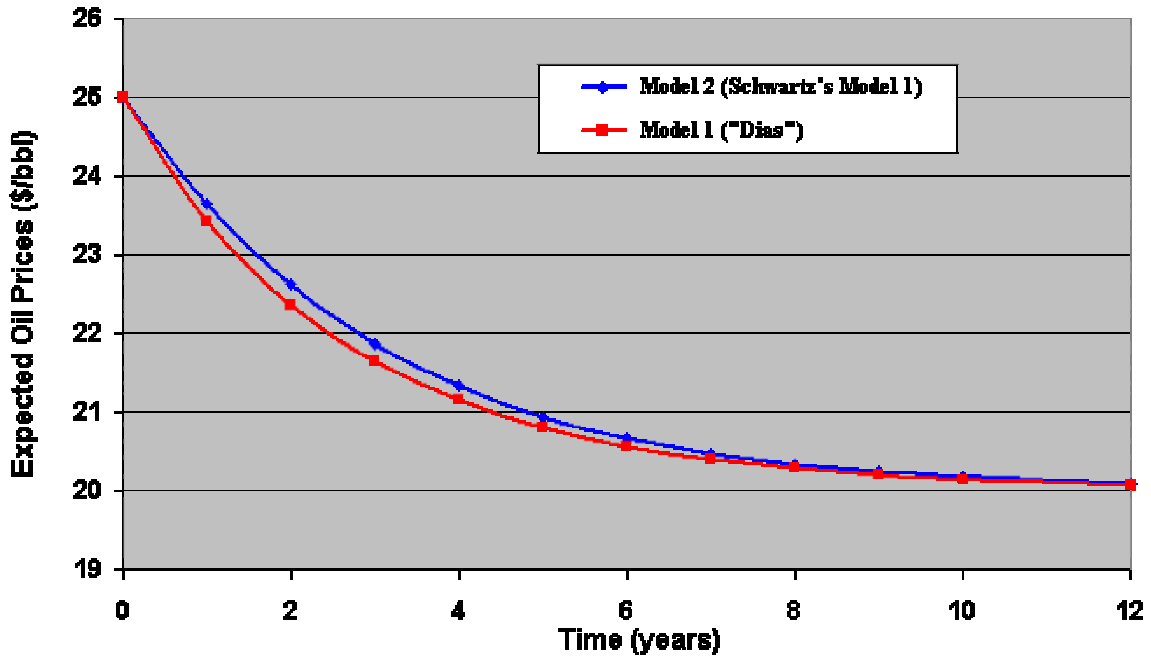
See more about the differences between risk-neutral process and real drift process for the mean-reversion case in the [FAQ13](#).

However, I prefer the model 1 than model 2 because is more simple to see where the simulation

is converging, and the formula of prices expectation is also more simple. Anyway, simulations has been showing (from a PUC-Petrobras project) that the real options results are very similar for the two models if both models are simulated with the same long-run level of convergence (by making the Schwartz's long-run futures price equal our Model 1 long-run equilibrium  $\bar{P}$ ) and the same reversion speed  $\eta$ , for example as in the chart below.

### **Comparison of Mean Reverting Expected Value Curves for Oil Prices**

**Starting Oil Price = 25 \$/bbl; Long-Run Equilibrium Level = 20 \$/bbl**



### **A More Practical Approach to Mean Reversion Model 2**

The main practical drawbacks from the Model 2 are: (a) the real simulation does not converges to the real equilibrium level  $\bar{P}$ ; and (b) the long-run convergence level from the real simulation depends on both volatility and reversion speed. See the equation of the "Long-Run Expected Futures Price" above that we repeat below for convenience.

$$F = E[P(\infty)] = \text{Expected Long-Run Futures Price} = \exp[\ln \bar{P} - (\sigma^2/(4\eta))] = \bar{P} \exp[-\sigma^2/(4\eta)]$$

The practical idea for the user is: instead using the equilibrium level  $\bar{P}$  as input, we enter as input the expected long-run futures price that the simulation converges. So, the equilibrium value is found by inverting the above formula:

$$\bar{P} = \exp[\ln F + (\sigma^2/(4\eta))] = F \exp[\sigma^2/(4\eta)]$$

Alternatively, we estimate  $\bar{P}$  from with a regression (see the page on stochastic processes) or other method, but we report to managers the value which the real simulation converges (F),

which is much more relevant in practice than  $\bar{P}$ .

In this way, the simulation equation for the **real** prices process is given by:

$$P_t = \exp\{ [\ln(P_{t-1}) (\exp(-\eta \Delta t))] + [(\ln F - (\sigma^2/(4\eta))) (1 - \exp(-\eta \Delta t))] + [\sigma (\text{SQRT}((1 - \exp(-2\eta \Delta t))/(2\eta))) N(0,1)] \}$$

Where the input **F** has the managerial meaning of *average long-run level that the simulation converges*.

For the risk-neutral simulation, the equation using **F** as input is obtained easily: just subtract from  $\ln F$  the *normalized risk-premium* in a similar way done before. This is left as exercise for the reader.

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## Monte Carlo Simulation of Mean Reversion with Jumps

This model, which I call *Marlim Model* (reference to Marlim, the top producer oilfield in Brazil), uses the first mean-reversion model presented before, but with the addition of random jumps - modeled with a Poisson process. In addition, jumps are of random size. The jump process  $dq$  is assumed to be independent of the continuous stochastic increment  $dz$ . Initially consider the following Arithmetic Ornstein-Uhlenbeck process with discrete jumps (modeled as a Poisson process) for a stochastic variable  $x(t)$ :

$$dx = \eta (\bar{x} - x) dt + \sigma dz + dq$$

This means that there is a reversion force over the variable  $x$  pulling towards an equilibrium level  $\bar{x}$ , like a spring force. The velocity of the reversion process is given by the parameter  $\eta$ . Jumps are represented by the term  $dq$ , which most of time is zero and sometimes occur jumps of uncertain size  $\phi$  and with arrival rate  $\lambda$ .

$$dq = 0 \text{ with probability } 1 - \lambda dt$$

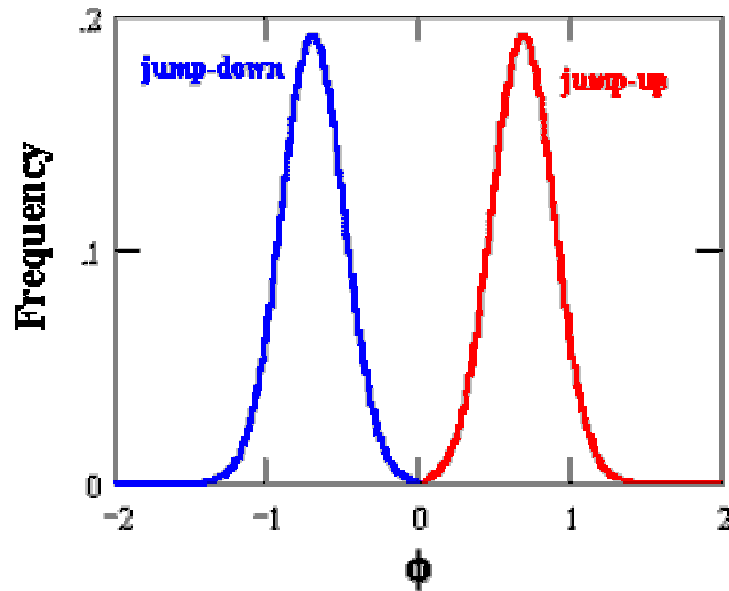
$$dq = \phi \text{ with probability } \lambda dt$$

The jump process  **$dq$  is assumed independent of  $dz$** , the Wiener increment from the continuous process. That is, the Poisson process is independent of the mean-reverting process.

The uncertain size of the jumps are modeled with the probability distribution  $\phi$ . In the classic paper of Merton (1976) of a jump-diffusion process, the jump-size distribution is assumed log-normal (so that the jumps occur in only one direction).

Here is allowed the possibility of both jumps-up and jumps-down, given that a jump occurred. Things are easier if we assume that there is the same probability of jump-up and jump-down, that

is, with a frequency 1/2 occurs jump-up and with a frequency 1/2 occurs jump-down. Let assume the following *symmetric* distributions for the jumps-up and jumps-down: a two Normal (truncated at zero, at least), with expected values of  $\ln(2)$  ( $\approx 0.693$ ) in case of jump-up and  $\ln(0.5)$  (or  $-\ln 2$ ) in case of jump-down. In addition, we assume equal probability for jump-up and jump-down, so that the expected jump-size is zero (in this case we don't need to use the [compensated Poisson process](#), because the expected value of the  $x(t)$  is already independent of the jump). See the figure below.



The variable  $x(t)$  has the following expressions for the mean and variance (see for example Das, 1998, *Poisson-Gaussian Processes and the Bond Markets*) at a future instant  $T$  (remember the expected jump-size in this case is zero):

$$\mathbf{E}[x(T)] = x(0) e^{-\eta T} + \bar{x} (1 - e^{-\eta T})$$

$$\mathbf{Var}[x(T)] = (1 - e^{-2\eta T}) \cdot \frac{(\sigma^2 + \lambda \cdot \mathbf{E}[\phi^2])}{2\eta}$$

The probability distribution of  $x(t)$  is NOT Normal (the jumps adds fatter tails to the Normal distribution from the mean-reversion component). For the higher moments expressions, see the previously mentioned paper of Das (1998).

In the expected value equation again the mean is just a weighted average between the initial level  $x(0)$  and the long-run level  $\bar{x}$  (the weights sum one). Note that there is no jump term in the expected value equation. This is the result of our jump model with symmetric distributions for the jumps up and down AND the 50% probability both jump directions in case of a jump. In the variance equation the jump term appear enlarging the variance when compared with the pure mean-reversion model 1.

It is important to remember that  $\mathbf{E}[\phi^2] \neq (\mathbf{E}[\phi])^2$ , and in order to obtain this value is necessary to

estimate the integral:

$$\mathbf{E}[\phi^2] = \int \phi^2 \cdot f(\phi) d\phi$$

Where  $f(\phi)$  is the probability density function (in the present case the two Normal densities). This integral depends of the distribution  $\phi$  but not of the jump arrival parameter  $\lambda$  (so that in practice you can evaluate once the integral, if the jump size distribution is fixed).

The stochastic process for the commodity price  $\mathbf{P}(t)$  is chosen so that the commodity prices are *function of  $\mathbf{x}(t)$*  described by the first equation of this item.

First let us to set the following relation between the long run process mean  $\bar{\mathbf{x}}$  and the long-run equilibrium price of the commodity  $\bar{\mathbf{P}}$ :

$$\bar{\mathbf{x}} = \ln(\bar{\mathbf{P}})$$

The idea is to set the prices in a way that the mean become  $\mathbf{E}[\mathbf{P}(T)] = e^{\mathbf{E}[\mathbf{x}(T)]}$ , that is, the relation between the variable  $x$  and  $P$  is such that in the simulation we get the following expected value for the commodity price at the instant  $T$ :

$$\mathbf{E}[\mathbf{P}(T)] = \exp\{\mathbf{x}(0) e^{-\eta T} + \bar{\mathbf{x}} (1 - e^{-\eta T})\}$$

The direct process  $P(t) = e^{x(t)}$  doesn't work here because the exponential of a normal distribution adds the half of the variance in the log-normal distribution mean. In order to reach the goal ( $\mathbf{E}[\mathbf{P}(T)] = e^{\mathbf{E}[\mathbf{x}(T)]}$ ), the half of the variance is compensated using the equation:

$$\mathbf{P}(t) = \exp\{\mathbf{x}(t) - 0.5 \text{Var}[\mathbf{x}(t)]\}$$

Where  $\text{Var}[\mathbf{x}(t)]$  is a deterministic function of time, including the jumps contribution, given before in this topic. In the literature (for example Fu et al., 2001), the half-variances of both the continuous process and jump process are subtracted in the simulation equation, but separately.

In the *risk-neutral* format, the process  $\mathbf{x}(t)$  is simulated using the discrete-time expression with an *appropriate* time-step  $\Delta t$ :

$$\mathbf{x}_t = \mathbf{x}_{t-1} e^{-\eta \Delta t} + [\ln(\bar{\mathbf{P}}) - ((\mu - r)/\eta)] (1 - e^{-\eta \Delta t}) + \sigma \sqrt{(1 - \exp(-2\eta \Delta t))/(2\eta)} N(0,1) + \text{jumps}$$

Where  $\mu$  is the risk-adjusted discount rate for the underlying asset  $P$ . The red term  $(\mu - r)/\eta$  is the *normalized risk-premium* subtracting the long run mean level  $\bar{\mathbf{x}} (= \ln(\bar{\mathbf{P}}))$ .

The simulation process is simple. Calculate  $\mathbf{x}(t)$  with the equation above, sampling both the standard Normal distribution  $N(0, 1)$  and the possibility of jump: Poisson process with parameter  $\lambda_{\text{up}}$  and  $\lambda_{\text{down}}$  for jumps-up and jumps-down, respectively.

Most of time the Poisson sampling points no jumps ( $dq = 0$ ), but in case of jumps, we need to sampling the Normal distributions either for jumps-up or jumps-down.

Now, with the simulated values of  $\mathbf{x}(t)$ , the simulation of  $\mathbf{P}(t)$  is easy. Use equation of price  $\mathbf{P}(t) = \exp\{\mathbf{x}(t) - 0.5 \text{Var}[\mathbf{x}(t)]\}$  together the equation for the variance of  $\mathbf{x}(t)$  in order to get  $\mathbf{P}(t)$ .

The term "jumps" in the equation above adds variance to the  $x(T)$  distribution. In case of a deterministic mean-reversion ( $\sigma = 0$ ) with stochastic jumps, the jumps effect looks significant. The jumps effect on the variance of  $x(T)$  that most of time obeys a deterministic mean-reverting process is:

$$\text{Var}[x(T)_{\sigma=0}] = (1 - e^{-2\eta T}) \cdot \frac{\lambda \cdot E[\phi^2]}{2\eta}$$

This variance initially grows but stabilizes for very long time  $T$ , due to the effect of the mean-reversion force. It is not difficult to see in the above equation that when, in addition to the  $\sigma = 0$ , the reversion speed  $\eta$  tends to zero (no mean-reversion) the **pure jumps process with random jump-size** given by the probability distribution  $\phi$ , has the following variance (take the limit in the last equation):

$$\text{Var}[x(T)_{\text{pure jumps}}] = \lambda E[\phi^2] T$$

Without the mean-reverting force, the variance of jumps grows with the time so that it is not bounded as before.

What is the **appropriate time-step  $\Delta t$** ? Although the discretization for the mean-reverting part of the equation of  $x(t)$  is exact (valid for very large  $\Delta t$ ), the presence of jumps together with the reversion creates some problems to use large  $\Delta t$ . For geometric Brownian process combined with jumps there is no problem because the process drift doesn't depend on the current level of the stochastic variable (it is possible even to use *Brownian bridge* with independent simulations for each process). The same is not true for mean-reversion process - the drift is function of the current value of the stochastic variable. Hence, the combination of reversion with jumps deserves more caution.

For the case with jumps combined with mean-reversion, I recommend to use a *relatively* small  $\Delta t$ , but larger than the required by Euler and Milstein approximations to reach a similar accuracy, in most cases.

The idea is as follow. If the time interval  $\Delta t$  is *sufficiently small*, the probability of two jumps occurrence is negligible because  $(\lambda \cdot \Delta t)^2$  is much lower than  $(\lambda \cdot \Delta t)$ . So in this case we can consider only one jump in each time interval.

In order to illustrate the problem, imagine for example a  $\Delta t = 1$  year and  $\lambda = 1/\text{year}$ . The probability of occurrence of two jumps in one year is given by  $P[N(t) = n] = (1/n!) (\lambda t)^n e^{-\lambda t} = 18.4\%$  (make  $n = 2$ ;  $\lambda = 1$ ;  $t = 1$ ), which is not negligible. If we consider both jumps at the ending of the year, the mean-reversion will not act through the year, reducing the jumps chocks effect at the ending of the year. A very different result will occur if one jump is considered at the beginning of the year (so the reversion will exercise a strong force along the year, and the other one at the ending of the year).

However, in the oil price process we are interested in rare large jumps only and we can use smaller time-steps. Assume  $\lambda = 0.25$  per annum (is expected only one jump each 4 years). By using a  $\Delta t = 1$  month (= 0.08 year), the probability of two jumps in this period (one month) is only 0.02%, which is negligible (the probability of one jump is > 2% and is not negligible). In practice this means that we can assume only one jump using a smaller  $\Delta t$  (as in this case).

Even for the case with jumps associated with the mean-reversion, is preferable to use the above discretization for  $x(t)$  than approximations for the mean-reverting part using Euler or Milstein approximations. The Euler or Milstein approximations require very small  $\Delta t$  even without jumps. With jumps, a  $\Delta t$  of one month in the above example means a larger error than using the exact discretization used for the mean-reverting part.

For the **real process simulation** (instead the risk-neutral one), use the same equation for  $x(t)$ , but without the *normalized risk-premium*  $(\mu - r)/\eta$  subtracting the long run mean level  $\bar{x}(= \ln(\bar{P}))$ , that is the real simulation for  $x(t)$  is given by:

$$x_t = x_{t-1} e^{-\eta \Delta t} + \ln(\bar{P}) (1 - e^{-\eta \Delta t}) + \sigma \sqrt{(1 - \exp(-2 \eta \Delta t)) / (2 \eta)} N(0,1) + \text{jumps}$$

The Excel spreadsheet available to **download** below, shows the real and risk-neutral sample-paths from a simulation of the mean-reversion with jumps, named Marlim model (**press F9 to get new sample paths in the chart**):

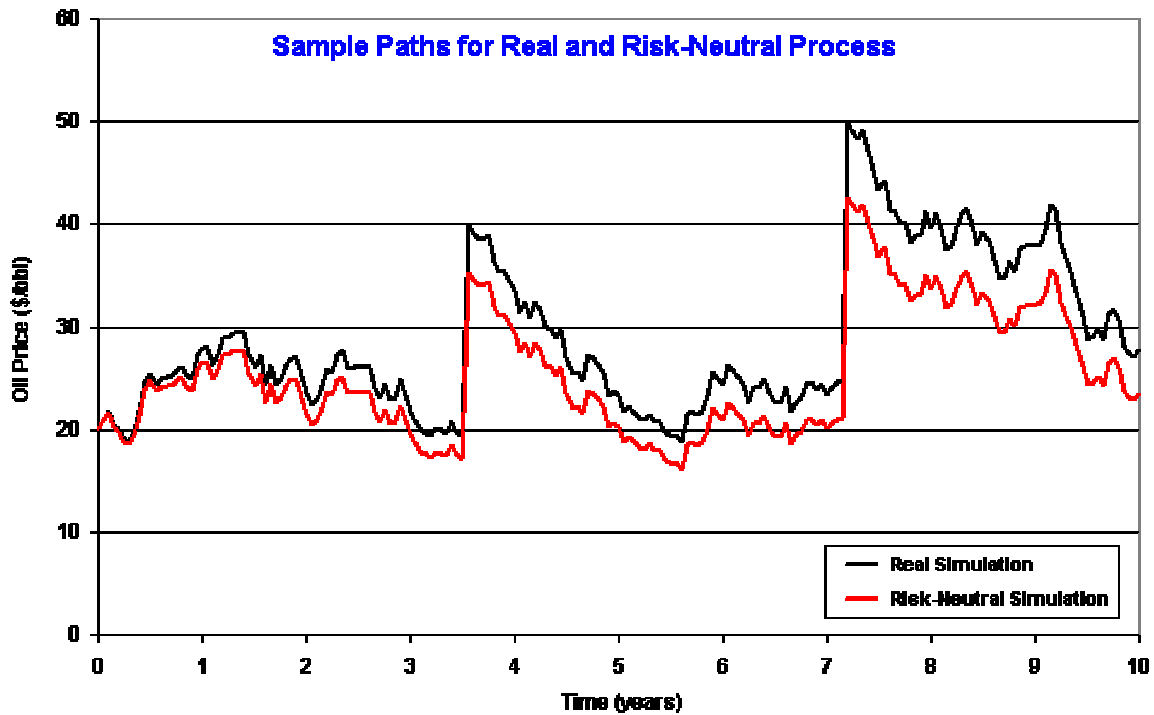
◆Download the Excel spreadsheet [simulation-reversion-jumps-marlim-real x rn.xls](#), with 235 KB

In the spreadsheet, were used a fixed jump-sizes for both jump-up and jump-down. However, it is easy to set probability distributions for the jumps-size. The formulas used in the spreadsheet are the ones presented above.

The chart below presents an example of sample paths simulation for both the real and the risk-neutral simulation of a mean-reversion with jumps, from the spreadsheet.

In this example the Marlim model considered  $\lambda = 0.25$  (is expected one jump each 4 years), with equal probability to be jump-up or jump-down; reversion with half-life of two years towards an equilibrium level of 20 \$/bbl; volatility of 20%, etc (see the spreadsheet available before for details).

### Simulation of Mean-Reversion with Jumps - Marlim Model



**Algorithm:** Most Monte Carlo commercial software (such as @Risk, Crystal Ball, etc.) have direct facilities for simulation of Poisson distributions. However, some readers could be interested on build up a simple simulator for Poisson distribution, and it is presented below. One algorithm for the simulation of jumps occurrence is given by the following simple procedure based in Dupire's "Monte Carlo Toolkit" (1998), and it is valid for small  $\Delta t$  (maximum one jump in this small interval):

- For each time-step, from 1 to n, with length  $\Delta t$ , draw a number  $u$  uniformly:  $u \sim U[0, 1]$ ;
- If  $u < \lambda \cdot \Delta t$ , a jump occurs and is necessary to draw a jump-size from the distribution of jumps-size  $\phi$ ;
- Else (if  $u \geq \lambda \cdot \Delta t$ ), set jumps = zero (no jump occurred in this time-interval) and go to the next time-interval.

**Many jumps in a time-interval-** If the time interval  $\Delta t$  is sufficiently small, the probability of two jumps occurrence is negligible. However, for large  $\Delta t$ , this is not true: is necessary to consider more than one jump. In this case the jumps term become:

$$\mathbf{jumps} = \sum_{i=1}^{J(\Delta t)} \phi_i$$

Where  $J(\Delta t) \sim \text{Poisson}(\lambda \cdot \Delta t)$  gives the number of jumps in this time interval, each of them with uncertain size  $\phi_i$ .



In case of no jumps, in the equation above jumps = 0.

In practical terms, make jumps = jumps-up + jumps-down, so we get two summations, one for jumps-up and another for jumps-down, each with arrival rate of  $\lambda/2$  (if jump-up and jump-down have 50% chances each in case of one jump) and with distributions  $\phi_{up}$  and  $\phi_{down}$ , respectively inside the summations. However, again I recommend a small  $\Delta t$  so that the probability of more than one jump is negligible.

In order to consider more than one jump in each time-interval, it is necessary a more general algorithm for Poisson processes. This algorithm is generally based in the relationship between the Poisson( $\lambda$ ) distribution and the exponential distribution,  $\text{expo}(1/\lambda)$ , is given in the textbook of Law & Kelton (2000, p.478).

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